A Multisector “AK Model” with Endogenous Growth: Existence and Characterization of Optimal Paths and Steady States Analysis

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Abstract. We study a multisector 'AK model' in continuous time that is intrinsically unbounded, nonsmooth, and non-strictly concave. We find an existence result for optimal strategies and a set of duality results (using the Maximum Principle and Dynamic Programming in an integrated way) and provide a complete classification of the price supported steady states of the model. We interpret these results with reference to such concepts as the Non-substitution theorem and the own rates of returns, already present in the literature on multisector models developed in the 50s and 60s on the basis of von Neumann's growth model.

Key words: Endogenous growth, AK model, optimal control with mixed constraints, optimality conditions, steady states, Non-substitution Theorem, von Neumann growth model.

1 Introduction

The linear growth model, which became prominent as the "AK model" with Rebelo [27], has rightly been dubbed "the simplest endogenous growth model" ([5, p. 38] (see also pp. 39-42 and 141-4 in the same book). Its characteristic feature is that there is only one commodity, whose production function has the form Y = AK, where Y is the output and K is the input, both consisting of quantities of the same commodity, and A is a positive constant reflecting the level of technological knowledge. With a capital good that is produced by itself, unbounded paths of production become feasible.

In this paper we generalize the AK model to the case of any (finite) number of commodities, assuming in accordance with the general thrust of the original model that all inputs are themselves producible. Moreover, we shall adopt the standard assumption in much of the modern literature on endogenous growth that there is an immortal representative agent who is concerned with maximizing an intertemporal utility function over an infinite time horizon. More precisely, it is assumed that the agent's instantaneous utility is actualized at a constant discount rate and is a function of the consumption of a single commodity with a constant elasticity of substitution between present and future consumption. Hence there are a number of capital goods but only one consumption good.

Rebelo [27] considered also a model in which the consumption good is produced by scarce resources and a capital good which is selfreproduced: if the factors available in fixed supply can be continuously and completely substituted, then the supply of the consumption good can be expanded indefinitely. In this paper we ignore scarce resources. And we do so not only in the sense that all commodities are technologically producible, but also in the sense that the amounts of commodities available at time zero are either positive or, if at time zero the amount of some commodity is zero, the parameters concerning consumption preferences are in ranges in which the long period consumption plans are not constrained by the non existence of such a commodity. We will refer to this condition as "full reproducibility". We also say that "scarcity does not enter in an essential way".

The AK model has been chosen since it is simple and yet can be said to convey one of the main messages of the new growth models: if there are a number of commodities which can reproduce themselves, with no contribution from other resources, then the growth process can be sustained even without technical progress. However, the reproducibility assumption has the far reaching implication that strict concavity is not a generic property of the model. Moreover, the fact that the smoothness property is strongly limited both with respect to production and with respect to consumption has also revealed a number of interesting properties of the model. With limited smoothness the "corner solution" is the rule rather than the exception. This enables us to clarify some aspects of the theory which are otherwise hidden.

As in the von Neumann model [24] the rule of free goods has been introduced; however the simple introduction of the rule of free goods with continuous time would have implied an infinite speed of disposal: we are faced here with the simple fact that disposal requires time. From an economic point of view it would be desirable to use a model that allows for lags in the relation of outputs to inputs in continuous time. But this would have transformed the model so much that its relationship with the literature would have been very restricted. The solution we use in this paper is the introduction of a finite rate of depreciation of commodities when they are disposed of, not lower than the rate of depreciation of the same commodities when they are used in production. Hence a more appropriate name would be that of “rule of free services of goods”. Clearly there is no need for a rule of free goods if there exists only one commodity.

These remarks give incidentally some indications how this paper is connected also to the von Neumann growth model and to the literature which developed in the sixties on Turnpike Theorems (for a survey, see [34, Chapter 7], see also [22]). This is not surprising since the AK model could be dated back decades before [27]: see, for example, the discussion in [34, Section 5.D.b]. In particular, when we study the steady states of the model we can recognize many properties known since the sixties or the seventies. The main differences from the von Neumann model itself are that here we do not allow joint production and, on the contrary, we allow consumption. But that literature studied these conditions too. Conversely, the main difference from the literature developed in the seventies on the Ramsey model consists in the fact that in the present model, as in the von Neumann model, no restrictions on growth come from constraints on the availability of "natural" factors; that is the model is not bounded.

The model is studied as an optimal control problem in $\mathbb{R}^n$ with $(2n+1)$ linear state-control constraints. Such a high number of constraints gives rise to non-smoothness in the problem; more precisely, the current value Hamiltonian has a component which is not a smooth function on the set of feasible data (see Section 4 for details). Moreover, since we are dealing with an unbounded growth model, we are naturally driven to consider unbounded consumption-production strategies, which creates a lack of compactness in the problem making it technically more difficult. As far as we know, models of the type under consideration are not treated in the mathematical literature on optimal control (e.g. [28, 29] treat general constraints but assume bounded strategies, and similarly [32], while [4] treats unbounded strategies but different types of constraints, and so on). Thus we were forced to study the problem from the beginning, trying to adapt the main methods of optimal control theory (Dynamic Programming , see e.g. [6], and Maximum Principle, see e.g. [26]) to our case. We stress the fact that to get our optimality conditions and to deduce from them the main features of the optimal trajectories we need to combine the two approaches above in an original way.

To be more precise, in this paper we provide:

- an existence result for optimal strategies;
- a set of necessary conditions and one of sufficient conditions;
- an analysis of the steady states of the model, with a complete classification of the steady states involving full reproducibility.

These results are proved under some restrictions (explained in Subsection 2.3) that we introduce to deal both with the complexity and the economic interpretation of the model. A full dynamic analysis of the model forms part of some further research.

Since we do not deal with the stability of the equilibrium paths, the long section on the classification of the steady states requires some further motivation, which will be provided at the very beginning of Section 6. Here it suffices to remark that steady states allow one to study the static and comparative static properties of the model. Moreover, when there is full reproducibility our duality results allow us to endow these states with stationary relative prices and therefore to define the "real rate of profit" associated with steady growth paths. One result is that steady states exist under the same conditions that yield existence of optimal strategies. Another is that three different types of steady states are envisaged. They depend on the size of the discount rate. Subsection 6.5 is devoted to providing an interpretation of these results. (When there is not full reproducibility, other optimal steady states can be found, but they are not price supported: two examples are provided in Appendix D.)

The main technical point to address has been the proof of necessary and sufficient optimality conditions. To prove them here we combine the two main tools for treating optimal control problems: the Maximum Principle and the Dynamic Programming (see Section 5). For the reader's convenience the main technical points are concentrated in the appendices that can all be skipped at a first reading. The main difficulty in dealing with the steady states has been related to the fact that whereas steady states are defined as equilibrium paths in which consumption and process intensities grow at a common rate, the stocks of available commodities do not need to grow at that rate (although the stocks of commodities with a positive price need to do so).

The content of the paper is the following.

- In Section 2 we describe the model. This section is divided in three parts: Subsection 2.1 where we describe the equation for the capital stock, Subsection 2.2 where we set out the optimal control problem of utility maximization, Subsection 2.3 where the main assumptions of the model are presented and explained.

- Section 3 is devoted to stating a result on the existence of optimal strategies that is proved in Appendix A.

- Section 4 contains a description of the Hamiltonians of our problem, pointing out their main features and properties and why we cannot apply standard techniques to deal with it.

- In Section 5 a set of necessary and a set of sufficient optimality conditions for our problem are stated. The proofs are provided in Appendix C, which uses the key results of the Dynamic Programming approach applied to our model contained in Appendix B. Some useful properties of optimal strategies are also provided.

- Section 6 contains the classification of steady states and their interpretation. It is divided in 5 subsections: the first three devoted to the definitions of steady states (6.1) and some preliminary results concerning admissible steady states (6.2) and price supported steady states (6.3); the fourth devoted to the classification theorem (6.4); the fifth devoted to the interpretation of such results (6.5). In subsection (6.1) we distinguish between optimal steady states which are price supported and optimal steady states which are not so. The latter may exist only if the condition of full reproducibility does not hold and for this reason they are not fully analyzed in this paper . Appendix D shows, by examples, that the set of optimal steady states which are not price supported is not empty.

2 The Model

There are n commodities, but only one of them is consumed, say commodity 1. Preferences with respect to consumption over time are such that they can be described by a single intertemporal utility function U_σ, which is the usual C.E.S. (Constant Elasticity of Substitution) function used in this kind of literature: for a given consumption path $c : [0, +\infty) \mapsto \mathbb{R}$ we set

$$U_\sigma (c(\cdot)) = \int_0^{+\infty} e^{-\rho t} u_\sigma (c(t))\, dt$$

where the instantaneous utility function u_σ depend on a given parameter $\sigma > 0$ (the so-called elasticity of substitution) and is given by

$$u_\sigma (c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \qquad for \quad \sigma > 0,\ \sigma \neq 1 \tag{1}$$

$$u_1 (c) = \log c \qquad for \quad \sigma = 1$$

(with the agreement that $u_\sigma(0) = -\infty$ for $\sigma \geq 1$) and $\rho \in \mathbb{R}$ is the rate of time preference (or discount rate) of the immortal representative agent. For the sake of simplicity we will drop the constant $-(1-\sigma)^{-1}$ in the following since this will not affect the optimal paths. We observe that most of the results of this paper hold also for more general utility functions u but we will not go in this direction to avoid technicalities.

Technology is fully described by an $n \times n$ material input matrix $\mathbf{A}$, the corresponding output matrix $\mathbf{I}$ (which is the $n \times n$ identity matrix), and by a uniform rate of depreciation $\delta_{\mathbf{x}}$ of capital goods used for production. The rate of depreciation for goods not employed in production is $\delta_{\mathbf{z}}$. Finally, no primary factor is used in production and there is no choice of technique. To simplify the technical problems and to concentrate on the most interesting features of the model we will make some further assumptions that will be described later (Subsection 2.3). Let us now describe the equation for the stock of capital, starting, for the sake of clarity, with the discrete time case. Then, in Subsections 2.2 and 2.3 we will give precise statement of our problem and assumptions.

2.1 The equation for the stock

Let us first formulate the problem in the discrete time case. This is done in order to specify some features of the continuous time model used in the paper. At time $t_0 = 0$ there is a starting amount of commodities $\mathbf{s}_0 = \bar{\mathbf{s}}$. This amount is partly used for producing outputs, partly it is disposed of so we have $\mathbf{s}_0^T = \mathbf{x}_0^T \mathbf{A} + \mathbf{z}_0^T$ (where $\mathbf{x}$ denotes the vector of the intensities of operation and $\mathbf{z}$ the amount of goods which are disposed of). At the end of the first period we have that the amount of commodities available $\mathbf{s}_1$ is equal to the amount of commodities produced for the first period, minus the amount consumed in that period, plus the residuals from both production and disposal activities:

$$\mathbf{s}_1 = \mathbf{x}_0^T \mathbf{I} + (1-\delta_{\mathbf{x}})\mathbf{x}_0^T \mathbf{A} + (1-\delta_{\mathbf{z}})\mathbf{z}_0^T - c_0 \mathbf{e}_1^T.$$

Moreover, the allocation in the next period gives

$$\mathbf{s}_1^T = \mathbf{x}_1^T \mathbf{A} + \mathbf{z}_1^T$$

So, repeating the argument for every period we have that the evolution is given by

$$\mathbf{s}_{t+1}^T = \mathbf{x}_t^T \mathbf{I} + (1-\delta_{\mathbf{x}})\mathbf{x}_t^T \mathbf{A} + (1-\delta_{\mathbf{z}})\mathbf{z}_t^T - c_t \mathbf{e}_1^T;$$

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \mathbf{z}_t^T.$$

Subtracting $\mathbf{s}_t$ the first equation becomes

$$\mathbf{s}_{t+1}^T - \mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{I} - \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A} - \delta_{\mathbf{z}} \mathbf{z}_t^T - c_t \mathbf{e}_1^T. \tag{2}$$

The initial datum is $\mathbf{s}_0 = \bar{\mathbf{s}}$ and the constraints are:

$$\mathbf{x}_t, \mathbf{z}_t \geq \mathbf{0}; \qquad \mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \mathbf{z}_t^T; \qquad c_t \geq 0. \tag{3}$$

Now let us go to the continuous time case. We replace the difference equation (2) and the constraints (3) by their continuous-time analogues. We then consider the following differential equation for the evolution of the commodities' stocks (corresponding to (2))

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T \left[\mathbf{I} - \delta_{\mathbf{x}} \mathbf{A}\right] - \delta_{\mathbf{z}} \mathbf{z}_t^T - c_t \mathbf{e}_1^T; \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

with the constraints (corresponding to (3))

$$\mathbf{x}_t, \mathbf{z}_t \geq \mathbf{0}; \qquad \mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \mathbf{z}_t^T; \qquad c_t \geq 0.$$

These inequalities and equations can be rewritten by eliminating the variable $\mathbf{z}$ and setting $\delta = -\delta_{\mathbf{z}} + \delta_{\mathbf{x}}$ as

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T \left[\mathbf{I} - \delta \mathbf{A}\right] - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T; \tag{4}$$

with the initial condition

$$\mathbf{s}_0 = \bar{\mathbf{s}}$$

and the constraints

$$\mathbf{x}_t \geq \mathbf{0}; \qquad \mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}; \qquad c_t \geq 0, \tag{5}$$

where $\mathbf{s}$ is the state variable and $\mathbf{x}$ and c are the control variables. From now on we will work with the setting (4) and (5).

Suppose that the production-consumption strategy $(\mathbf{x}, c)$ is a measurable and locally integrable function : $\mathbb{R}^+ \mapsto \mathbb{R}^n \times \mathbb{R}$ (we will denote by $L^1_{\text{loc}}(0, +\infty; \mathbb{R}^{n+1})$ the set of such functions). Then the differential equation (4) has a unique solution : $\mathbb{R}^+ \mapsto \mathbb{R}^n$ which is absolutely continuous (we will denote by $W^{1,1}_{\text{loc}}(0, +\infty; \mathbb{R}^n)$ the set of such functions). Such a solution will be denoted by the symbol $\mathbf{s}_{t;\bar{\mathbf{s}},(\mathbf{x},c)}$, omitting the subscript $\bar{\mathbf{s}},(\mathbf{x},c)$ when it is clear from the context. Of course such solution can be written in integral form as

$$\mathbf{s}_t^T = e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_s^T\left[\mathbf{I}-\delta\mathbf{A}\right]ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}c_s\mathbf{e}_1^T ds.$$

Given an initial endowment $\bar{\mathbf{s}}$ we will say that a strategy $(\mathbf{x},c) \in L^1_{\mathrm{loc}}\left(0,+\infty;\mathbb{R}^{n+1}\right)$ is admissible from $\bar{\mathbf{s}}$ if the triple $\left(\mathbf{x},c,\mathbf{s}_{t;\bar{\mathbf{s}},(\mathbf{x},c)}\right)$ satisfies the constraints (5). The set of admissible control strategies starting at $\bar{\mathbf{s}}$ will be denoted by $\mathcal{A}(\bar{\mathbf{s}})$. We stress the fact that, due to the constraints (5), the set of admissible control strategies depends on the initial endowment $\bar{\mathbf{s}}$.

We observe that the state variable $\mathbf{s}$ and the first control variable $\mathbf{x}$ appear only in the state equation and not in the functional to maximize. In any case, when talking about control strategies we will always refer to the pair $(\mathbf{x},c)$ (unless clearly specified). Due to the constraint $\mathbf{s}_t^T \geq \mathbf{x}_t^T\mathbf{A}$ and to the semipositivity of $\mathbf{A}$ the set $\mathcal{A}(\bar{\mathbf{s}})$ is clearly a subset of the space $L^\infty_{loc}\left(0,+\infty;\mathbb{R}^n\right) \times L^1_{loc}\left(0,+\infty;\mathbb{R}\right)$ (where we denote by $L^\infty_{loc}\left(0,+\infty;\mathbb{R}^n\right)$ the set of locally bounded functions: $\mathbb{R} \mapsto \mathbb{R}^n$). We deal with such a general set of control strategies (instead of the usual set of piecewise continuous strategies) because our existence result holds only in such context (which is a natural feature of all main existence results).

In some cases we will make use of a truncated problem which is obtained by deleting some row and some column of matrix $\mathbf{A}$ and the corresponding rows and columns of matrix $\mathbf{I}$. In this case the state equation (4) becomes

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T\left[\mathbf{D}-\delta\mathbf{C}\right] - \delta_{\mathbf{z}}\mathbf{s}_t^T - c_t\mathbf{e}_1^T, \tag{6}$$

where $\mathbf{D}$ and $\mathbf{C}$ are the truncated matrices (from matrices $\mathbf{I}$ and $\mathbf{A}$, respectively).

2.2 The optimal control problem

Given $\rho \in \mathbb{R}$, $\sigma > 0$ and the instantaneous utility u_σ as in (1) (dropping for simplicity the constant $(1-\sigma)^{-1}$ when $\sigma \neq 1$), we fix the initial endowment $\bar{\mathbf{s}}$ and we consider the problem (P_σ) of maximizing the future discounted utility

$$U_\sigma(c) = \int_0^{+\infty} e^{-\rho t}u_\sigma\left(c\left(t\right)\right)dt$$

over all production-consumption strategies $(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}})$.

Definition 2.1 *A strategy* $(\mathbf{x}^*,c^*) \in \mathcal{A}(\bar{\mathbf{s}})$ *will be called optimal if we have* $U_\sigma(c^*) > -\infty$ *and*

$$+\infty > U_\sigma(c^*) \geq U_\sigma(c)$$

for every admissible control pair $(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}})$.

A weaker definition of optimality can be used in the case in which we allow for strategies that takes values $+\infty$ (or when all strategies take value $-\infty$). We do not treat this case in this paper; see [32, Section 3.7] for precise definitions.

We finally define the value function as

$$V(\bar{\mathbf{s}}) = \sup_{(\mathbf{x},c)\in\mathcal{A}(\bar{\mathbf{s}})} U_\sigma(c) \tag{7}$$

recalling that the properties of V will be studied in Appendix B and that they will be used to establish necessary conditions of optimality (in particular to establish the necessity of the so-called transversality condition, as in e.g. [7]) and to study the steady states.

2.3 Main Assumptions

We now list and comment on the assumptions we will make troughout the paper.

Assumption 2.2 $\mathbf{A}$ *is a* $n \times n$ *matrix, nonnegative and irreducible.*

This assumption means that in this paper we are setting on one side the problem of choice of techniques. The assumption on irreducibility of matrix $\mathbf{A}$ is made for the sake of simplicity. It means that each commodity enters directly or indirectly into the production of any other commodity. The case of a reducible matrix $\mathbf{A}$ is interesting both for the economist and the mathematician. It will be dealt with in another paper.

The eigenvalue of $\mathbf{A}$ with maximum modulus will be denoted by $\lambda_{PF} > 0$. We call $\mathbf{v}_{PF}$ the right eigenvector relative to λ_{PF} such that $\mathbf{e}_1^T\mathbf{v}_{PF} = 1$ and $\mathbf{s}_{PF}$ the left eigenvector relative to λ_{PF} such that $\mathbf{s}_{PF}^T\mathbf{e}_1 = 1$. We remember that $\mathbf{v}_{PF} > \mathbf{0}$, $\mathbf{s}_{PF} > \mathbf{0}$, by the Frobenius Theorem (see [20, pp. 509-519], [34, pp. 367-380]).

Assumption 2.3 $1 - \delta_{\mathbf{x}}\lambda_{PF} \geq 0$

This assumption implies that technology allows the economy to grow at a uniform nonnegative rate. Without it the analysis would be more general, but less interesting.

The two assumptions 2.2 and 2.3 will be always made. In some case, to go deeper in the study of the main properties of the model, we will use also the following four assumptions.

Assumption 2.4 $\delta \leq 0$ *i.e.* $\delta_{\mathbf{x}} \leq \delta_{\mathbf{z}}$.

This means that a commodity used in production decades less quickly (or at most at the same rate) than the same commodity when is not used in production. This corresponds to the idea of disposal. The case of positive δ would correspond to the idea of conservation. But then there is a problem of choice of technique between production and conservation. This case will be studied in another paper, devoted to the choice of technique.

Assumption 2.5 $\bar{\mathbf{s}} > \mathbf{0}$.

This means that at time $t = 0$ the stock of every capital good is positive: it is done to ensure reproducibility and to avoid a surrepticious introduction of scarcity. An example could clarify this point. If $a_{jj} > 0$ and the j-th element of $\bar{\mathbf{s}}$ is nought, then the j-th commodity cannot be produced for each $t \geq 0$. The model is obviously equivalent to the one in which in the state equation matrices $\mathbf{I}$ and $\mathbf{A}$ are substituted with matrix $\mathbf{D}$ and $\mathbf{C}$, where matrix $\mathbf{C}$ is obtained from $\mathbf{A}$ by deleting the j-th column and all rows of $\mathbf{A}$ which on the j-th column have a positive element (the j-th row is among the deleted rows) and matrix $\mathbf{D}$ is obtained from $\mathbf{I}$ by deleting the corresponding rows and the j-th column. The commodities which exist and are produced by the processes depicted by the deleted rows are in this way forced to be a sort of exhaustible resources, which are available, but cannot be produced.

In analyzing steady states we take account also of some cases with semipositive $\bar{\mathbf{s}}$, since in steady states analysis $\bar{\mathbf{s}}$ is endogenously determined. This will be done, however, only in cases in which scarcity does not enter in an essential way (in a sense which will be clarified later).

Assumption 2.6 $a_{11} > 0$.

This means that the consumption good enters in its own production: it is in fact a strong assumption from the economic point of view; however without this assumption the mathematical treatment of the model would be much more complicated due to the presence of jumps in co-state variables which yield necessary conditions in terms of finitely additive measures (see on this [25, Theorem VI.3.93]).

Assumptions 2.5 and 2.6 imply that the optimal trajectory is positive and that the co-state variables are absolutely continuous (see Appendices B and C, Subsection C.2). We are currently working on an extension of the model to the more general case when Assumption 2.6 fails.

Assumption 2.7 $a := \rho - \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1-\sigma) > 0$.

When assumption 2.5 holds this is equivalent to assume existence of optimal strategies with finite utility (see Theorem 3.1 below).

3 Existence of optimal strategies

In this section we study the problem of existence of optimal strategies. We have the following result.

Theorem 3.1 *Let Assumptions 2.2, 2.3 and 2.5 hold true. For any $\sigma > 0$ and initial datum $\bar{\mathbf{s}} > \mathbf{0}$, there is an optimal strategy $(\mathbf{x}, c)$ for problem (P_σ) if and only if Assumption 2.7 holds. Moreover this strategy is unique in the sense that, if $(\hat{\mathbf{x}}, \hat{c})$ is another optimal strategy, then $\hat{c} = c$. If Assumption 2.7 does not hold then:*

(i) if $\sigma \in (0,1)$ then there is an admissible strategy with utility $+\infty$;

(ii) if $\sigma = 1$ and $\lambda_{PF}^{-1} - \delta_{\mathbf{x}} > 0$ then there is an admissible strategy with utility $+\infty$;

(iii) if $\sigma = 1$ and $\lambda_{PF}^{-1} - \delta_{\mathbf{x}} = 0$ then every admissible strategy has utility $-\infty$;

(iv) if $\sigma \in (1, +\infty)$ then every admissible strategy has utility $-\infty$.

Proof. See Appendix A. ■

Remark 3.2 In the proof of Theorem 3.1 the critical step is finding the subset of the parameters' space for which the value function if finite. This is done by providing some estimates on the growth rate of all admissible strategies (to prove that V is finite or $-\infty$) and/or some examples (to prove that V is $+\infty$). In each case the needed restriction is given by Assumption 2.7. Once the value function is finite, well-known compactness techniques apply, so that existence is granted.

The full proof requires some technicalities and a lot of calculations that are relegated to Appendix A. ■

In fact the above theorem could be extended also to the case when $\bar{\mathbf{s}} \geq \mathbf{0}$, with suitable adjustments. We skip them for brevity. From now on we will assume that Assumption 2.7 holds.

In the next two Sections we supplement Theorem 3.1 with a set of duality results. In particular, we introduce the (current) shadow prices for the constraints (4) and (5) and prove a set of sufficient conditions and a set of necessary conditions involving such shadow prices. The economic interpretation concerns the existence of a price path supporting the optimal path of capital accumulation.

4 The Hamiltonians

In next section we will provide necessary and sufficient conditions for an admissible solution to be optimal by using Hamiltonians. The present section show some special features of the Hamiltonians of the problem at hand and why we cannot use known results.

The current value Hamiltonian H of our problem (P_σ) is given, for $\sigma \neq 1$, by

$$H(\mathbf{s}, \mathbf{v}; \mathbf{x}, c) = -\delta_{\mathbf{z}} \mathbf{s}^T \mathbf{v} + \mathbf{x}^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{v} - c \mathbf{e}_1^T \mathbf{v} + \frac{c^{1-\sigma}}{1-\sigma} \qquad \mathbf{s}, \mathbf{v}, \mathbf{x}, \in \mathbb{R}^n;$$

$$c \in [0, +\infty), \text{ if } \sigma < 1, \text{ and } c \in (0, +\infty), \text{ if } \sigma > 1$$

and, for $\sigma = 1$ by

$$\begin{aligned} H(\mathbf{s}, \mathbf{v}; \mathbf{x}, c) &= -\delta_{\mathbf{z}} \mathbf{s}^T \mathbf{v} + \mathbf{x}^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{v} - c \mathbf{e}_1^T \mathbf{v} + \log c \\ \mathbf{s}, \mathbf{v}, \mathbf{x}, &\in \mathbb{R}^n; \quad c \in (0, +\infty). \end{aligned}$$

Note that it is the sum of three parts:

$$H_1(\mathbf{s}, \mathbf{v}) = -\delta_{\mathbf{z}} \mathbf{s}^T \mathbf{v}; \qquad H_2(\mathbf{v}; \mathbf{x}) = \mathbf{x}^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{v}$$

$$H_3(\mathbf{v};c) = -c\mathbf{e}_1^T\mathbf{v} + \frac{c^{1-\sigma}}{1-\sigma}; \qquad \text{or} \qquad -c\mathbf{e}_1^T\mathbf{v} + \log c$$

where H_1 does not depend on the control $(\mathbf{x}, c)$.

The maximum value Hamiltonian is, for $\mathbf{s}, \mathbf{v} \in \mathbb{R}^n$ if $\sigma < 1$

$$H_0(\mathbf{s},\mathbf{v}) = \max_{(\mathbf{x},c)\geq 0,\ \mathbf{x}^T\mathbf{A}\leq \mathbf{s}^T} H(\mathbf{s},\mathbf{v};\mathbf{x},c)$$

while, if $\sigma \geq 1$

$$H_0(\mathbf{s},\mathbf{v}) = \max_{(\mathbf{x},c)\geq 0,\ \mathbf{x}^T\mathbf{A}\leq \mathbf{s}^T} H(\mathbf{s},\mathbf{v};\mathbf{x},c)$$

(note that we use the notation max instead of sup since we know that the maximum is attained here). If we have

$$\mathbf{e}_1^T\mathbf{v} > 0$$

then the maximum point of $H_3(\mathbf{v};c)$ is attained at $c = \left(\mathbf{e}_1^T\mathbf{v}\right)^{-1/\sigma}$, so that, for $\sigma \neq 1$,

$$H_0(\mathbf{s},\mathbf{v}) = -\delta_{\mathbf{z}}\mathbf{s}^T\mathbf{v} + \max_{\mathbf{x}\geq 0,\mathbf{x}^T\mathbf{A}\leq \mathbf{s}^T}\{\mathbf{x}^T[\mathbf{I}-\delta\mathbf{A}]\mathbf{v}\} + \frac{\sigma}{1-\sigma}\left(\mathbf{e}_1^T\mathbf{v}\right)^{\frac{\sigma-1}{\sigma}}.$$

and, for $\sigma = 1$

$$H_0(\mathbf{s},\mathbf{v}) = -\delta_{\mathbf{z}}\mathbf{s}^T\mathbf{v} + \max_{\mathbf{x}\geq 0,\mathbf{x}^T\mathbf{A}\leq \mathbf{s}^T}\{\mathbf{x}^T[\mathbf{I}-\delta\mathbf{A}]\mathbf{v}\} - 1 - \log\left(\mathbf{e}_1^T\mathbf{v}\right)$$

On the contrary, if

$$\mathbf{e}_1^T\mathbf{v} = 0$$

then for $\sigma \in (0,1]$

$$H_0(\mathbf{s},\mathbf{v}) = +\infty.$$

while, for $\sigma > 1$

$$H_0(\mathbf{s},\mathbf{v}) = -\delta_{\mathbf{z}}\mathbf{s}^T\mathbf{v} + \max_{\mathbf{x}\geq 0,\mathbf{x}^T\mathbf{A}\leq \mathbf{s}^T}\{\mathbf{x}^T[\mathbf{I}-\delta\mathbf{A}]\mathbf{v}\}.$$

Finally, if

$$\mathbf{e}_1^T\mathbf{v} < 0$$

then, for every $\sigma > 0$,

$$H_0(\mathbf{s},\mathbf{v}) = +\infty.$$

To simplify notation we define:

$$\begin{aligned} H_{01}(\mathbf{s},\mathbf{v}) &= -\delta_{\mathbf{z}}\mathbf{s}^T\mathbf{v}; \qquad H_{02}(\mathbf{s},\mathbf{v}) = \max_{\mathbf{x}\geq 0,\mathbf{x}^T\mathbf{A}\leq \mathbf{s}^T}\{\mathbf{x}^T[\mathbf{I}-\delta\mathbf{A}]\mathbf{v}\} \\ H_{03}(\mathbf{v}) &= \frac{\sigma}{1-\sigma}\left(\mathbf{e}_1^T\mathbf{v}\right)^{\frac{\sigma-1}{\sigma}} \ \text{or} \ -1-\log\left(\mathbf{e}_1^T\mathbf{v}\right), \end{aligned}$$

so that, for $\sigma \in (0,1]$

$$H_0(\mathbf{s},\mathbf{v}) = \begin{cases} H_{01}(\mathbf{s},\mathbf{v}) + H_{02}(\mathbf{s},\mathbf{v}) + H_{03}(\mathbf{v}); & \text{if } \mathbf{e}_1^T\mathbf{v} > 0 \\ +\infty; & \text{if } \mathbf{e}_1^T\mathbf{v} \leq 0 \end{cases}$$

and, for $\sigma > 1$

$$H_0(\mathbf{s},\mathbf{v}) = \begin{cases} H_{01}(\mathbf{s},\mathbf{v}) + H_{02}(\mathbf{s},\mathbf{v}) + H_{03}(\mathbf{v}); & \text{if } \mathbf{e}_1^T\mathbf{v} > 0 \\ H_{01}(\mathbf{s},\mathbf{v}) + H_{02}(\mathbf{s},\mathbf{v}); & \text{if } \mathbf{e}_1^T\mathbf{v} = 0 \\ +\infty; & \text{if } \mathbf{e}_1^T\mathbf{v} < 0 \end{cases}.$$

Recall that $H_{02}(\mathbf{s},\mathbf{v})$ is the value of the maximum of the linear programming problem:

$$\begin{cases} \max \quad \mathbf{x}^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v} \\ \mathbf{x}^T\mathbf{A} \le \mathbf{s}^T \\ \mathbf{x} \ge \mathbf{0} \end{cases} \tag{8}$$

The corresponding dual problem is

$$\begin{cases} \min \quad \mathbf{s}^T\mathbf{q} \\ \mathbf{A}\mathbf{q} \ge [\mathbf{I} - \delta\mathbf{A}]\mathbf{v} \\ \mathbf{q} \ge \mathbf{0}. \end{cases} \tag{9}$$

Since both problems have feasible solutions (also due to the semi-positivity of each row of matrix $\mathbf{A}$) then both problems have optimal solutions and

$$\bar{\mathbf{x}} \in \text{argmax}\,\{\mathbf{x}^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v}_t;\ \mathbf{x} \ge \mathbf{0};\ \mathbf{x}^T\mathbf{A} \le \mathbf{s}^T\}$$

if and only if there exists $\bar{\mathbf{q}} \in \mathbb{R}^n$ such that

$$\mathbf{A}\bar{\mathbf{q}} \ge [\mathbf{I} - \delta\mathbf{A}]\mathbf{v}; \quad \bar{\mathbf{q}} \ge 0; \quad \bar{\mathbf{x}}^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v} = \bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{q}} = \mathbf{s}^T\bar{\mathbf{q}}.$$

This will be used in the formulation of optimality conditions in the next section.

Note that the Hamiltonians H_2 (with respect to production) and H_3 (with respect to consumption) present some unpleasant features from the technical point of view:

- H_2 is not strictly convex so the maximum point does not need to be unique: moreover the set of optimal solutions (which give the optimal production strategy) is not a smooth function on the set of feasible data (usually this is associated with a lack of uniqueness for the optimal strategy $\mathbf{x}$ and with a lack of regularity for the value function and for the dual variable $\mathbf{v}$).
- The domain of H_2 depends on the state variable (due to the state - control constraint).
- Both H_2 and H_3 can be infinite for some value of the dual variables $(\mathbf{v},\mathbf{q})$.
- The domain of H_3 is not bounded.

As far as we know, models of the type discussed are not treated in the mathematical literature on optimal control (e.g. [28, 29] treat general constraints but assume bounded strategies, and similarly [32], while [4] treats unbounded strategies but different types of constraints, and so on). This is the reason, mentioned in the introduction, which forced us to study the problem since from the beginning.

5 Optimality Conditions

In this section we discuss optimality conditions for our problems. Then we will use them to study the nature of the steady state solutions in Section 6. The main result are Theorem 5.1 and Theorem 5.3 below where a set of sufficient and a set of necessary conditions is stated.

Let us begin with the sufficient conditions.

Theorem 5.1 (Sufficient conditions). *Let Assumption 2.2, 2.3 and 2.7 hold. If* $(\hat{\mathbf{x}}, \hat{c})$ *is an admissible production-consumption strategy starting at* $\bar{\mathbf{s}}$, $\hat{\mathbf{s}}$ *is the associated commodities' stock trajectory and there exist two functions* $\mathbf{v}, \mathbf{q} : \mathbb{R}^+ \mapsto \mathbb{R}^n$ *such that* $\mathbf{v}$ *is absolutely continuous,* $\mathbf{q}$ *is measurable and locally integrable and they satisfy for almost every* $t \geq 0$*:*

$$\dot{\hat{\mathbf{s}}}_t^T = -\delta_{\mathbf{z}} \hat{\mathbf{s}}_t^T + \hat{\mathbf{x}}_t^T [\mathbf{I} - \delta \mathbf{A}] - \hat{c}_t \mathbf{e}_1^T \tag{10}$$

$$\dot{\mathbf{v}}_t = (\rho + \delta_{\mathbf{z}}) \mathbf{v}_t - \mathbf{q}_t; \tag{11}$$

$$\hat{c}_t^{-\sigma} = \mathbf{e}_1^T \mathbf{v}_t > 0 \tag{12}$$

$$(\mathbf{I} - \delta \mathbf{A}) \mathbf{v}_t \leq \mathbf{A} \mathbf{q}_t \tag{13}$$

$$\hat{\mathbf{x}}_t \geq \mathbf{0}; \qquad \hat{\mathbf{x}}_t^T \mathbf{A} \leq \hat{\mathbf{s}}_t^T; \qquad \mathbf{q}_t \geq \mathbf{0} \tag{14}$$

$$\hat{\mathbf{x}}_t^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{v}_t = \hat{\mathbf{x}}_t^T \mathbf{A} \mathbf{q}_t = \hat{\mathbf{s}}_t^T \mathbf{q}_t \tag{15}$$

$$\lim_{t \to +\infty} e^{-\rho t} \hat{\mathbf{s}}_t^T \mathbf{v}_t = 0 \tag{16}$$

$$\mathbf{v}_t \geq \mathbf{0} \tag{17}$$

Then $(\hat{\mathbf{x}}, \hat{c})$ *is optimal.*

Proof. See Appendix C. ■

Remark 5.2 In fact the above Theorem 5.1 holds under more general Assumptions. In particular we mention the following extensions, which come straightforwardly from the proof, which will be needed in studying the steady states in Section 6 and in Appendix D.

- The condition (17) can be substituted with the weaker one:

  $$\lim_{t \to +\infty} e^{-\rho t} \mathbf{s}_t^T \mathbf{v}_t \geq 0$$

  for every admissible trajectory $\mathbf{s}$ starting at $\bar{\mathbf{s}}$.

- The matrix $\mathbf{I}$ in the state equation can be substituted by a semipositive matrix $\mathbf{B}$; moreover, both $\mathbf{A}$ and $\mathbf{B}$ can be rectangular (and congruent) and $\mathbf{A}$ does not need to be irreducible. Finally Assumption 2.3 does not need to hold.

■

We now pass to necessary conditions.

Theorem 5.3 (Necessary conditions). *Let Assumptions 2.2, 2.3, 2.5, 2.6 and 2.7 hold. Let* $(\hat{\mathbf{x}}, \hat{c})$ *be an optimal production-consumption strategy starting at* $\bar{\mathbf{s}}$, $\hat{\mathbf{s}}$ *be the associated commodities' stock trajectory. Then there exists two functions* $\mathbf{v}, \mathbf{q} : \mathbb{R}^+ \mapsto \mathbb{R}^n$ *such that* $\mathbf{v}$ *is absolutely continuous,* $\mathbf{q}$ *is measurable and locally bounded and they satisfy for almost every* $t \geq 0$*:*

$$\dot{\hat{\mathbf{s}}}_t^T = -\delta_{\mathbf{z}} \hat{\mathbf{s}}_t^T + \hat{\mathbf{x}}_t^T [\mathbf{I} - \delta \mathbf{A}] - \hat{c}_t \mathbf{e}_1^T \tag{18}$$

$$\dot{\mathbf{v}}_t = (\rho + \delta_{\mathbf{z}}) \mathbf{v}_t - \mathbf{q}_t; \tag{19}$$

$$\hat{c}_t^{-\sigma} = \mathbf{e}_1^T \mathbf{v}_t > 0 \tag{20}$$

$$(\mathbf{I} - \delta \mathbf{A}) \mathbf{v}_t \leq \mathbf{A} \mathbf{q}_t \tag{21}$$

$$\hat{\mathbf{x}}_t \geq \mathbf{0}; \qquad \hat{\mathbf{x}}_t^T \mathbf{A} \leq \hat{\mathbf{s}}_t^T; \qquad \mathbf{q}_t \geq \mathbf{0} \tag{22}$$

$$\hat{\mathbf{x}}_t^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{v}_t = \hat{\mathbf{x}}_t^T \mathbf{A} \mathbf{q}_t = \hat{\mathbf{s}}_t^T \mathbf{q}_t \tag{23}$$

$$\lim_{t \to +\infty} e^{-\rho t} \hat{\mathbf{s}}_t^T \mathbf{v}_t = 0 \tag{24}$$

Moreover for a.e. $t \geq 0$*,*

$$\mathbf{v}_t \in D^+ V (\hat{\mathbf{s}}_t) \tag{25}$$

(where $D^+ V (\mathbf{s})$ *denotes the superdifferential of* V *at a point* $\mathbf{s}$*).*

Proof. See Appendix C. ∎

Remark 5.4 In the case when $\bar{\mathbf{s}} \geq 0$ The above Theorem 5.3 holds in a weaker form. In particular the locally bounded function $\mathbf{q}$ becomes a finitely additive measure and the co-state inclusion (25) can fail to hold. ∎

Remark 5.5 Observe that for a.e. $t \geq 0$ the point $\hat{\mathbf{x}}_t$ is a solution of the linear programming problem (8) with $\mathbf{s} = \hat{\mathbf{s}}_t$ and $\mathbf{v} = \hat{\mathbf{v}}_t$. Moreover the point $\hat{c}_t$ is the solution of the optimization problem $\max\{-c\mathbf{e}_1^T \mathbf{v}_t + u_\sigma(c)\}$. This is the so-called Maximum Principle. Observe also that $\mathbf{q}$ is a solution of the dual problem (9) with $\mathbf{s} = \hat{\mathbf{s}}_t$ and $\mathbf{v} = \hat{\mathbf{v}}_t$. ∎

It is well known that if a complete set of markets exists, then the prices supporting the optimal path are also competitive equilibrium prices. In this respect equations and inequalities (18)-(24) can be interpreted as if at each instant of time there are both spot markets (in which the 'services' of capital stocks are traded) and asset markets (that allow agents to transfer purchasing power through time); moreover, the consumer owns the whole capital, which she continuously rents to 'firms' on competitive spot markets, and (continuously) trades on the asset markets in order to save (or dissave). In this interpretation, $\mathbf{q}_t$ is the vector of spot rental rates for the capital goods and $\mathbf{v}_t$ is the vector of spot prices of stocks. We also note that in view of Theorem 5.3, one can reinterpret Theorem 3.1 as an existence (and uniqueness) result for competitive equilibria for an economy so depicted. In the following we will refer to $\mathbf{v}_t$ as the (shadow) prices and to $\mathbf{q}_t$ as the (shadow) rentals.

Due to the "nonsmooth" nature of the problem the proofs of Theorems 5.1 and 5.3 are not contained in known results that one can refer to. We will give them, at least for the parts where we cannot apply directly standard results given in the literature. We remark that the proof of necessity is quite heavy and strongly uses results of Appendix B where we apply the dynamic programming method to our problem.

Remark 5.6 We point out the following technical facts about Theorem 5.3.

1. The co-state inclusion (25) is quite hard to obtain but it plays a key role here. In fact, using the properties of the value function V proved in Appendix B it allows one to get the nonnegativity of $\mathbf{v}$ and the transversality condition (24) and these are key points to prove the classification theorem in Section 6.

2. The transversality condition (24) is in general not necessary for infinite horizon problems. It turns out to be necessary here due to the concavity of the problem. To prove the necessity of it we use the co-state inclusion (25) and the properties of the value function proved in Appendix B in particular its concavity (see [7], for similar arguments).

3. In general the co-state $\mathbf{v}$ can be discontinuous. However, in our case assumptions 2.5 and 2.6 rule out the discontinuity of $\mathbf{v}$ since they guarantee that a special kind of "constraints qualification" holds in our case (see [25, Hypothesis VI.3.98]. More precisely such "constraints qualification" follows from the fact that, when starting from a positive $\bar{\mathbf{s}}$, the optimal trajectory remains positive. This fact is a consequence of the regularity of the value function and is proven in Corollary B.6 in Appendix B. See also [32], [31], [17]) for another, more classical, kind of "constraints qualification".

4. The lack of strict concavity in the Hamiltonian H_2 (i.e. no unique maximum point) is a source of nonsmoothness in our problem.

5. Assumption 2.4 is not needed here. ■

The above results allow us to prove the following properties of the optimal strategy $\hat{c}$ that will be useful in studying the steady states.

Proposition 5.7 *Let Assumptions 2.2 ,2.3, 2.5 and 2.7 hold. Assume also that* $(\hat{\mathbf{x}},\hat{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ *is optimal for the problem* (P_σ) *and let* $\hat{\mathbf{s}}$ *be the corresponding optimal state. Then the optimal consumption strategy* $\hat{c}$ *is absolutely continuous and strictly positive and satisfies the differential equation*

$$\dot{\hat{c}}_t = \frac{\hat{c}_t}{\sigma}\left[-\rho - \delta_{\mathbf{z}} + \frac{\mathbf{e}_1^T \mathbf{q}_t}{\mathbf{e}_1^T \mathbf{v}_t}\right]; \quad \hat{c}_0 = \left(\mathbf{e}_1^T \mathbf{v}_0\right)^{-1/\sigma}. \tag{26}$$

where $\mathbf{v}$ *and* $\mathbf{q}$ *are the co-state path introduced in Theorems 5.1 and 5.3. Moreover we have for every* $t \geq 0$

$$\hat{c}_t \geq \left(\mathbf{s}_{PF}^T \mathbf{v}_0\right)^{-1/\sigma} e^{(-1/\sigma)\left(\rho+\delta_{\mathbf{x}}-\lambda_{PF}^{-1}\right)t} \tag{27}$$

Proof. The absolute continuity of c and equation (26) easily follow from the necessary optimality condition (20) which gives

$$\hat{c}_t = \left(\mathbf{e}_1^T \mathbf{v}_t\right)^{-1/\sigma}$$

and from the co-state equation (19).

Moreover (27) follows from the inequality

$$\mathbf{s}_{PF}^T \mathbf{v}_t \leq e^{\left(\rho+\delta_{\mathbf{x}}-\lambda_{PF}^{-1}\right)t} \mathbf{s}_{PF}^T \mathbf{v}_0 \tag{28}$$

which we prove now. Taking the inner product of the inequality (21) by $\mathbf{s}_{PF}^T$ we obtain

$$\mathbf{A}\mathbf{q}_t - \left[\mathbf{I}-\delta\mathbf{A}\right]\mathbf{v}_t \geq 0 \Longrightarrow \mathbf{s}_{PF}^T \left[\mathbf{I}-\delta\mathbf{A}\right]\mathbf{v}_t \leq \mathbf{s}_{PF}^T \mathbf{A}\mathbf{q}_t$$

$$\Longleftrightarrow (1-\delta\lambda_{PF})\,\mathbf{s}_{PF}^T\mathbf{v}_t \leq \lambda_{PF}\mathbf{s}_{PF}^T\mathbf{q}_t \Longleftrightarrow -\mathbf{s}_{PF}^T\mathbf{q}_t \leq -\left(\lambda_{PF}^{-1}-\delta\right)\mathbf{s}_{PF}^T\mathbf{v}_t$$

so that

$$\mathbf{s}_{PF}^T\dot{\mathbf{v}}_t = (\rho+\delta_{\mathbf{z}})\,\mathbf{s}_{PF}^T\mathbf{v}_t - \mathbf{s}_{PF}^T\mathbf{q}_t \leq \left(\rho+\delta_{\mathbf{x}}-\lambda_{PF}^{-1}\right)\mathbf{s}_{PF}^T\mathbf{v}_t$$

and the claim (28) follows.

Now, substituting estimate (28) into (20) we get

$$\mathbf{e}_1^T\mathbf{v}_t \leq \mathbf{s}_{PF}^T\mathbf{v}_t \leq e^{\left(\rho+\delta_{\mathbf{x}}-\lambda_{PF}^{-1}\right)t}\mathbf{s}_{PF}^T\mathbf{v}_0$$

so that

$$c_t = \left(\mathbf{e}_1^T\mathbf{v}_t\right)^{-1/\sigma} \geq e^{(-1/\sigma)\left(\rho+\delta_{\mathbf{x}}-\lambda_{PF}^{-1}\right)t}\left(\mathbf{s}_{PF}^T\mathbf{v}_0\right)^{-1/\sigma}$$

and the claim follows. ■

6 Steady state solutions

In this section we study the optimal steady state solutions to problem (P_σ) providing the existence of optimal steady state solutions and further results regarding their structure and, for some values of the parameters, their uniqueness.

There are at least two reasons to study optimal steady state solutions. The most important one is certainly connected to the possibility of proving the convergence of the optimal path towards the steady state solution. In this context the study of the static and comparative static properties of the model sets the stage for the study of the dynamics of optimal paths. However, stability results are not reported here, because a preliminary examination of them showed that delicate points arise in the general case. Asymptotic turnpike theorems are known to hold for various discrete-time versions of the linear growth model ([2], [14], [19]). Moreover, it is common wisdom that at least the local version of the (asymptotic) turnpike result can be easily proved for 'low' values of the discount rate (see, [22]). Nevertheless, it turned out that the technical issues of non-smoothness and the lack of strict concavity that characterize our continuous-time framework preclude the use of known results even for the analysis of local stability. This convinced us that a full dynamic analysis of the model requires a specific study.

The other reason to deal with steady states is that in these states some relevant concepts, such as that of "real rate of profit" or "growth rate", can be defined. The literature on growth often refers to steady states in order to convey some macroeconomic insights. In [21], for example, such steady state concepts as the "rate of growth" or the "real rate of profit" that in general are meaningless with regard to non-stationary paths, are freely used under the explicit assumption of a fast convergence to the steady state. Section 6.5 is devoted to these analyses.

We should like to add that some growth theorists simply do not believe that the assumption of an infinitely lived and omniscient representative agent is appropriate to describe the behavior of a changing system. From this point of view, once the controversial assumption of perfect foresight is abandoned, the laws of change that characterize an optimal path become irrelevant. But such objection does not seem to apply to the set of paths with a stationary structure.

6.1 Definitions of steady state solutions

We start with the following definitions.

Definition 6.1 *A state-control pair* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ *of problem* (P_σ) *will be called a steady state solution (or simply a* steady state*) if it solves the state equation (4) and satisfies*

$$c_t = c_0 e^{gt}, \quad \mathbf{x}_t^T = \mathbf{x}_0^T e^{gt}, \quad \forall t \geq 0 \tag{29}$$

for a given constant rate of growth, $g \in \mathbb{R}$*,* $c_0 > 0$ *and* $\mathbf{x}_0 \geq \mathbf{0}$*.*

Definition 6.2 *A steady state solution* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ *of problem* (P_σ) *will be called an admissible steady state solution (or simply an* admissible steady state*) if it satisfies the constraints*

$$\mathbf{x}_t^T \mathbf{A} \leq \mathbf{s}_t^T, \qquad \forall t \geq 0.$$

Definition 6.3 *An admissible steady state solution* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ *of problem* (P_σ) *will be called an optimal steady state solution (or simply an* optimal steady state*) if it is an optimal pair for the problem* (P_σ)*.*

Definition 6.4 *An admissible steady state solution* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ *of problem* (P_σ) *will be called a price supported steady state solution (or simply a* price supported steady state*) if there exist two functions* $\mathbf{v}, \mathbf{q} : \mathbb{R}^+ \mapsto \mathbb{R}^n$ *such that* $\mathbf{v}$ *is absolutely continuous,* $\mathbf{q}$ *is measurable and locally bounded and they satisfy for almost every* $t \geq 0$*:*

$$\dot{\mathbf{v}}_t = (\rho + \delta_{\mathbf{z}})\, \mathbf{v}_t - \mathbf{q}_t \tag{30}$$

$$\hat{c}_t^{-\sigma} = \mathbf{e}_1^T \mathbf{v}_t > 0 \tag{31}$$

$$(\mathbf{I}-\delta\mathbf{A})\,\mathbf{v}_t \leq \mathbf{A}\mathbf{q}_t \tag{32}$$

$$\mathbf{q}_t \geq 0 \tag{33}$$

$$\mathbf{x}_t^T\,(\mathbf{I}-\delta\mathbf{A})\,\mathbf{v}_t = \mathbf{x}_t^T\mathbf{A}\mathbf{q}_t = \mathbf{s}_t^T\mathbf{q}_t \tag{34}$$

$$\lim_{t\to+\infty} e^{-\rho t}\hat{\mathbf{s}}_t^T\mathbf{v}_t = 0 \tag{35}$$

$$\mathbf{v}_t \in D^+V\,(\hat{\mathbf{s}}_t) \qquad \textit{for a.e. } t \geq 0. \tag{36}$$

Remark 6.5 We note that in a steady state solution the stock path $\mathbf{s}$ does not need to grow at a uniform rate, however the state equation

$$\dot{\mathbf{s}}_t^T = \left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right]e^{gt} - \delta_{\mathbf{z}}\mathbf{s}_t^T$$

strongly constrains its structure since it implies

$$\mathbf{s}_t^T = \frac{1}{g+\delta_{\mathbf{z}}}\left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right]e^{gt} + \left\{\mathbf{s}_0^T - \frac{1}{g+\delta_{\mathbf{z}}}\left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right]\right\}e^{-\delta_{\mathbf{z}}t}$$

if $g+\delta_{\mathbf{z}} \neq 0$, and

$$\mathbf{s}_t^T = \left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right]te^{-\delta_{\mathbf{z}}t} + \mathbf{s}_0^Te^{-\delta_{\mathbf{z}}t}$$

if $g+\delta_{\mathbf{z}} = 0$. ■

Remark 6.6 Theorem 5.1 ensures that every price supported steady state is optimal, while Theorem 5.3 ensures that an optimal steady state with $\bar{\mathbf{s}} > \mathbf{0}$ is also a price-supported steady state. The examples reported in Appendix D, however, show that there are optimal steady states with a nonpositive $\bar{\mathbf{s}}$ which are *not* price-supported. These steady states are obtained by superimposing that some commodity is not available at time 0 and, as a consequence of this fact and of the form of matrix $\mathbf{A}$, is not available at any time. That is, scarcity enters in an essential way. These cases are not to be confused with the cases in which, for some values of the parameters, the conditions mentioned in Definition 6.4 require that some commodities are not available at any time. Since this paper is devoted to the analysis of cases of full reproducibility, the optimal steady states which are not price supported are not fully studied in this paper (our conjecture is that such steady states play no role in the dynamical analysis of paths starting from positive $\bar{\mathbf{s}}$).

6.2 Admissibility of steady state solutions

According with Definition 6.2 and Remark 6.5 a steady state solution is admissible if and only if it satisfies, for every $t \geq 0$, the constraint

$$\mathbf{x}_0^T\mathbf{A} \leq \frac{1}{g+\delta_{\mathbf{z}}}\left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right] + \left\{\mathbf{s}_0^T - \frac{1}{g+\delta_{\mathbf{z}}}\left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right]\right\}e^{-(g+\delta_{\mathbf{z}})t} \tag{37}$$

if $g+\delta_{\mathbf{z}} \neq 0$, and

$$\mathbf{x}_0^T\mathbf{A} \leq \left[\mathbf{x}_0^T\,(\mathbf{I}-\delta\mathbf{A}) - c_0\mathbf{e}_1^T\right]t + \mathbf{s}_0^T \tag{38}$$

if $g+\delta_{\mathbf{z}} = 0$. Then the following result holds.

Proposition 6.7 *Let Assumption 2.2 hold. A steady state solution* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ *for problem* (P_σ) *is admissible if and only if*

$$\mathbf{x}_0^T \mathbf{A} \leq \mathbf{s}_0^T \tag{39}$$

and either

$$\mathbf{x}_0^T \left[\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq c_0 \mathbf{e}_1^T \quad \text{for} \quad g + \delta_{\mathbf{z}} \geq 0 \tag{40}$$

or

$$\mathbf{s}_0^T - \frac{1}{g + \delta_{\mathbf{z}}} \left[\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) - c_0 \mathbf{e}_1^T\right] \geq \mathbf{0} \quad \text{for} \quad g + \delta_{\mathbf{z}} \geq 0. \tag{41}$$

Moreover, the constraint $\mathbf{x}_t^T \mathbf{A} \leq \mathbf{s}_t^T$ *is binding (i.e. satisfied with equality) with regard the* $j-$*th component* for some $t > 0$ *if and only if it is binding* for every $t \geq 0$ *and this happens if and only if* $\mathbf{x}_0^T \mathbf{A} \mathbf{e}_j = \mathbf{s}_0^T \mathbf{e}_j$ *and*

$$\mathbf{x}_0^T \left[\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{A}\right] \mathbf{e}_j = c_0 \mathbf{e}_1^T \mathbf{e}_j \tag{42}$$

Remark 6.8 Observe that we always have $g + \delta_{\mathbf{x}} < \lambda_{PF}^{-1}$. In the case when $g + \delta_{\mathbf{x}} \geq 0$ this follows from (40), Assumption 2.4 and the Frobenius Theorem. In the case when $g + \delta_{\mathbf{x}} < 0$ this follows from the positivity of λ_{PF}^{-1}. ∎

Proof of Proposition 6.7. It is clear that admissibility always implies $\mathbf{x}_0^T \mathbf{A} \leq \mathbf{s}_0^T$. For the conditions (40) and (41) we have:

1. if $g + \delta_{\mathbf{z}} > 0$, then by letting $t \to +\infty$ (37) implies

   $$\mathbf{x}_0^T \mathbf{A} \leq \frac{1}{g + \delta_{\mathbf{z}}} \left[\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) - c_0 \mathbf{e}_1^T\right] \Longleftrightarrow \mathbf{x}_0^T \left[\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq c_0 \mathbf{e}_1^T.$$

   Vice versa, if (40) holds then

   $$\mathbf{x}_0^T \mathbf{A} \leq \frac{1}{g + \delta_{\mathbf{z}}} \left[\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) - c_0 \mathbf{e}_1^T\right]$$

   and this, plus $\mathbf{x}_0^T \mathbf{A} \leq \mathbf{s}_0^T$, gives that (37) holds.

2. if $g + \delta_{\mathbf{z}} = 0$, then, dividing (38) by t and letting $t \to +\infty$ we get

   $$\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) \geq c_0 \mathbf{e}_1^T. \tag{43}$$

   which is (40). Vice versa it is clear that (43) plus $\mathbf{x}_0^T \mathbf{A} \leq \mathbf{s}_0^T$ implies (38).

3. if $g + \delta_{\mathbf{z}} < 0$, multiplying (37) by $e^{(g+\delta_{\mathbf{z}})t}$ and letting $t \to +\infty$ we get that (41) holds. Vice versa it is immediately clear that (41) and $\mathbf{x}_0^T \mathbf{A} \leq \mathbf{s}_0^T$ imply (37).

We prove the final claim only for the case $g + \delta_{\mathbf{z}} > 0$ as the other cases are completely similar. In this case the right hand side of (37) is exponentially decreasing. So, if the constraint in (37) is binding for a given $t > 0$ in the j-th component, then the right hand side is constant. This gives exactly

$$\mathbf{x}_0^T \mathbf{A} \mathbf{e}_j = \frac{1}{g + \delta_{\mathbf{z}}} \left[\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) - c_0 \mathbf{e}_1^T\right] \mathbf{e}_j.$$

∎

6.3 Price supported steady state solutions and existence of price supports with constant growth rate

We study here the properties of the price-rental path for a given price supported steady state. In the general case $\bar{\mathbf{s}} \geq 0$ we get precise constraints on the data $g \in \mathbb{R}$, $c_0 > 0$, $\mathbf{x}_0 \geq 0$, on the stock path $\mathbf{s}_t$ and on the corresponding price-rental paths $(\mathbf{v}, \mathbf{q})$. In particular we get the existence of price supports with constant growth rate. Recall that, when $\bar{\mathbf{s}} \geq 0$ but not $\bar{\mathbf{s}} > 0$ the optimal state solutions do not need to be price supported (see Appendix D). The following result holds.

Proposition 6.9 *Let Assumptions 2.2, 2.3, 2.4, 2.6 and 2.7 hold. Then any price supported steady state solution* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ *of problem* (P_σ) *with* $\mathbf{s}_0 \geq 0$ *satisfies*

(i) $g \in \left[\sigma^{-1}\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \rho\right), \lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)$ *and this interval is always nonempty; as a consequence*

$$\rho - g(1-\sigma) > 0, \qquad \rho + \delta_{\mathbf{x}} + g\sigma > 0. \tag{44}$$

(ii) for every $t \geq 0$ *and for every price-rental path* $(\mathbf{v}_t, \mathbf{q}_t)$ *satisfying the conditions (30) – (36) mentioned in Definition 6.4*

$$\mathbf{e}_1^T \mathbf{v}_t = c_0^{-\sigma} e^{-g\sigma t}, \quad \mathbf{e}_1^T \mathbf{q}_t = (\rho + \delta_{\mathbf{z}} + \sigma g)\, c_0^{-\sigma} e^{-g\sigma t}, \tag{45}$$

$$\mathbf{x}_0^T \mathbf{A} \mathbf{e}_1 = \mathbf{s}_0^T \mathbf{e}_1, \qquad \mathbf{s}_t^T \mathbf{e}_1 = \mathbf{s}_0^T \mathbf{e}_1 e^{gt}, \tag{46}$$

$$\mathbf{x}_0^T \left[\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{A}\right] \mathbf{e}_1 = c_0, \tag{47}$$

$$\left[\mathbf{x}_0^T \left[\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{A}\right] - c_0 \mathbf{e}_1^T\right] \mathbf{e}_j \mathbf{e}_j^T \mathbf{q}_t = 0; \qquad \left[\mathbf{x}_0^T \mathbf{A} - \mathbf{s}_0^T\right] \mathbf{e}_j \mathbf{e}_j^T \mathbf{q}_t = 0 \qquad \forall j = 1 \ldots n. \tag{48}$$

(iii) there exists a price vector $\mathbf{v}_0 \geq 0$ *such that the price-rental path*

$$(\mathbf{v}_t, \mathbf{q}_t) = \left(\mathbf{v}_0 e^{-g\sigma t}, (\rho + \delta_{\mathbf{z}} + g\sigma)\, \mathbf{v}_0 e^{-g\sigma t}\right) \tag{49}$$

satisfies the conditions mentioned in definition 6.4. Such $(\mathbf{v}_t, \mathbf{q}_t)$ *will be called a* steady state price-rental path *supporting the steady state* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$.

Remark 6.10 From point (iii) we obtain that any admissible steady state solution with $g \in \left[\frac{1}{\sigma}\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \rho\right), \lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)$ is price supported if and only if there is a vector $\mathbf{v_0}$ such that:

$$\mathbf{e}_1^T \mathbf{v}_0 = c_0^{-\sigma} > 0 \tag{50}$$

$$\left[\mathbf{I} - (\rho + \delta_{\mathbf{x}} + g\sigma) \mathbf{A}\right] \mathbf{v}_0 \leq \mathbf{0}, \tag{51}$$

$$\mathbf{v}_0 \in D^+ V(\mathbf{s_0}), \tag{52}$$

$$\mathbf{x}_0^T \left[\mathbf{I} - (\rho + \delta_{\mathbf{x}} + g\sigma) \mathbf{A}\right] \mathbf{v}_0 = 0. \tag{53}$$

$$\left[\mathbf{x}_0^T \mathbf{A} - \mathbf{s}_0^T\right] \mathbf{v}_0 = 0. \tag{54}$$

■

Proof of Proposition 6.9.

Proof of (i). We observe first that the proof of Proposition 5.7 holds also when we are dealing with a price supported steady state. Then, by the statement of Proposition 5.7 we have, for suitable $M > 0$

$$c_t \geq M e^{\sigma^{-1}\left[\lambda_{PF}^{-1}-\rho-\delta_{\mathbf{x}}\right]}, \quad \forall t \geq 0$$

which gives $g \geq \sigma^{-1}\left[\lambda_{PF}^{-1} - \rho - \delta_{\mathbf{x}}\right]$. Moreover from Remark 6.8 we get the required upper bound.

Observe now that the interval $\left[\sigma^{-1}\left[\lambda_{PF}^{-1} - \rho - \delta_{\mathbf{x}}\right], \lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)$ is always nonempty since by Assumption 2.7 we have

$$\rho > \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1-\sigma) \Longleftrightarrow \sigma^{-1}\left[\lambda_{PF}^{-1} - \rho - \delta_{\mathbf{x}}\right] < \lambda_{PF}^{-1} - \delta_{\mathbf{x}}.$$

Finally we observe that inequalities (44) are obvious consequences of the bounds on g and of the positivity of λ_{PF}.

Proof of (ii). From equations (30) and (31) we have (45). In fact

$$\mathbf{e}_1^T \mathbf{v}_t = c_t^{-\sigma} = c_0^{-\sigma} e^{-g\sigma t}$$

$$\mathbf{e}_1^T \mathbf{q}_t = (\rho + \delta_{\mathbf{z}})\, \mathbf{e}_1^T \mathbf{v}_t - \mathbf{e}_1^T \dot{\mathbf{v}}_t = (\rho + \delta_{\mathbf{z}} + \sigma g)\, \mathbf{e}_1^T \mathbf{v}_t = (\rho + \delta_{\mathbf{z}} + \sigma g)\, c_0^{-\sigma} e^{-g\sigma t}.$$

Since $\rho + \delta_{\mathbf{z}} + \sigma g > 0$, $\mathbf{e}_1^T \mathbf{q}_t > 0$. Hence, from (34) we get

$$\mathbf{x}_t^T \mathbf{A} \mathbf{e}_1 = \mathbf{s}_t^T \mathbf{e}_1, \quad \forall t \geq 0.$$

And (46) follows by the definition of the steady state.

Now from the admissibility conditions (37) and (38) it follows that, on the first component

$$\mathbf{x}_0^T \mathbf{A} \mathbf{e}_1 = \frac{1}{g + \delta_{\mathbf{z}}}\left[\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) - c_0 \mathbf{e}_1^T\right] \mathbf{e}_1$$

when $g + \delta_{\mathbf{z}} \neq 0$ and

$$\mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A})\, \mathbf{e}_1 - c_0 = 0$$

when $g + \delta_{\mathbf{z}} = 0$ so that the claim (47) easily follows.

Finally (48) follows from (37) and from (34).

Proof of (iii).

Step 1. We observe that, for $j \in \{2, ..., n\}$ such that either

$$(g + \delta_{\mathbf{z}})\, \mathbf{s}_0^T \mathbf{e}_j - \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A})\, \mathbf{e}_j \neq 0$$

or

$$(g + \delta_{\mathbf{z}})\, \mathbf{s}_0^T \mathbf{e}_j - \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A})\, \mathbf{e}_j = 0 \qquad \textit{and} \qquad \mathbf{x}_0^T \mathbf{A} \mathbf{e}_j < \mathbf{s}_0^T \mathbf{e}_j$$

we get

$$\mathbf{e}_j^T \mathbf{v}_t = \mathbf{e}_j^T \mathbf{v}_0 = 0$$

for any price path $\mathbf{v}_t$.

In fact, from Proposition 6.7 in both cases the constraint $\mathbf{x}_t^T \mathbf{A} \mathbf{e}_j \leq \mathbf{s}_t^T \mathbf{e}_j$ is never binding for $t > 0$. This yields, thanks to condition (34) that $\mathbf{e}_j^T \mathbf{q}_t = 0$ for $t > 0$ for every rental path $\mathbf{q}$.

Then, from (30) we get $\mathbf{e}_j^T \mathbf{v}_t = e^{(\rho+\delta_{\mathbf{z}})t} \mathbf{e}_j^T \mathbf{v}_0$ for every price path $\mathbf{v}$. When $g + \delta_{\mathbf{z}} \neq 0$ the transversality condition (35) yields, setting $k_j = (g + \delta_{\mathbf{z}})^{-1} \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{e}_j$,

$$e^{-\rho t} \left(\mathbf{e}_j^T \mathbf{v}_t \right) \left(\mathbf{s}_t^T \mathbf{e}_j \right) = \mathbf{e}_j^T \mathbf{v}_0 \left[k_j e^{(\delta_z + g)t} + \left(\mathbf{s}_0^T \mathbf{e}_j - k_j \right) \right] \to 0, \quad as \quad t \to +\infty.$$

Similarly, when $g + \delta_{\mathbf{z}} = 0$, setting $k_j = \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{e}_j$, we have

$$e^{-\rho t} \left(\mathbf{e}_j^T \mathbf{v}_t \right) \left(\mathbf{s}_t^T \mathbf{e}_j \right) = \mathbf{e}_j^T \mathbf{v}_0 \left[k_j t + \left(\mathbf{s}_0^T \mathbf{e}_j \right) \right] \to 0, \quad as \quad t \to +\infty.$$

Now, if

$$(g + \delta_{\mathbf{z}}) \mathbf{s}_0^T \mathbf{e}_j - \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{e}_j \neq 0$$

the claim is immediate. If

$$(g + \delta_{\mathbf{z}}) \mathbf{s}_0^T \mathbf{e}_j - \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{e}_j = 0 \qquad and \qquad \mathbf{x}_0^T \mathbf{A} \mathbf{e}_j < \mathbf{s}_0^T \mathbf{e}_j$$

then for $g + \delta_{\mathbf{z}} = 0$ we get $k_j = 0$ but $\mathbf{s}_0^T \mathbf{e_j} > 0$ which gives the claim. For $g + \delta_{\mathbf{z}} > 0$ we observe that $\mathbf{s}_0^T \mathbf{e}_j = k_j > 0$ and the claim follows. Finally for the case $g + \delta_{\mathbf{z}} < 0$ we need to use Assumption 2.4. In fact, by $\mathbf{s}_0^T \mathbf{e}_j = k_j > 0$ we get that $\mathbf{x}_0^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{e}_j < 0$ which contradicts Assumption 2.4.

Step 2. Here we claim that

$$D_j^+ V (\mathbf{s}_t) = e^{-g\sigma t} D_j^+ V (\mathbf{s}_0) \qquad \forall j = 1, \ldots, n,$$

and that V is constant in all directions j as in the above step 1. The proof of these two facts follows by applying the regularity properties of V proved in Appendix B (Proposition B.2(iii)) and is omitted here for brevity.

Step 3. Finally take any admissible price path $\mathbf{v}_t$. Take its starting point $\mathbf{v}_0$ and set

$$\hat{\mathbf{v}}_t := e^{-g\sigma t} \mathbf{v}_0 \in D^+ V (\mathbf{s}_t) \quad \forall t \geq 0.$$

Then the path $\hat{\mathbf{v}}_t$, $\hat{\mathbf{q}}_t = (\rho + \delta_{\mathbf{z}} + g\sigma) \hat{\mathbf{v}}_t$ satisfies all sufficient conditions by a nontrivial verification procedure that uses the properties of superdifferentials and the concept of viscosity solution. We omit it for brevity.

Note that we only have to look at j such that

$$(g + \delta_{\mathbf{z}}) \mathbf{s}_0^T \mathbf{e}_j - \mathbf{x}_0^T (\mathbf{I} - \delta \mathbf{A}) \mathbf{e}_j = 0 \quad and \quad \mathbf{x}_0^T \mathbf{A} \mathbf{e}_j = \mathbf{s}_0^T \mathbf{e}_j.$$

■

6.4 The classification theorem

We now use the results of the previous subsections to prove existence and uniqueness of price supported steady states and to give a classification of them. We will show how the price supported steady states vary depending on the value of the discount rate ρ with respect to the other parameters of the model and show for one case, which turns out to be the only one with nonnegative rate of growth, uniqueness up to a multiplication by a constant.

We will see that the following three cases arise (recall that by Assumption 2.7 we have $\rho > \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1-\sigma)$)

- If $\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1-\sigma) < \rho < \lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1-\sigma)$ then there exists a unique (up to a multiplication by a constant) price supported steady state with

$$g = \sigma^{-1}\left[\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \rho\right] \in \left(-\delta_{\mathbf{x}}, \lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)$$

- If $\lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1-\sigma) \leq \rho \leq a_{11}^{-1} - \delta_{\mathbf{x}}(1-\sigma)$ then there exists a cone of dimension ≥ 1 of price supported steady states with growth rate $g = -\delta_{\mathbf{x}}$ classified in Theorem 6.11.

- If $\rho > a_{11}^{-1} - \delta_{\mathbf{x}}(1-\sigma)$ then there exists a cone of dimension n of price supported steady states with growth rate

$$g = \sigma^{-1}\left[a_{11}^{-1} - \delta_{\mathbf{x}} - \rho\right] \in (-\infty, -\delta_{\mathbf{x}})$$

classified in Theorem 6.11.

Summing up we can say that we always have uniqueness of the growth rate g (that will depend on the value of ρ with respect to the other parameters) while the uniqueness of the trajectory (up to a constant) holds only in the first case.

Theorem 6.11 *Let Assumptions 2.2, 2.3, 2.4, 2.6, and 2.7 hold. Then*

1. *If*

$$\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1-\sigma) < \rho < \lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1-\sigma),$$

then there exists a unique (up to a multiplication by a positive scalar constant) price supported steady state $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ of problem (P_σ) given by

$$\mathbf{s}_t^T = \mathbf{s}_0^T e^{gt}, \quad c_t = c_0 e^{gt}, \quad \mathbf{x}_t^T = \mathbf{x}_0^T e^{gt}, \quad \forall t \geq 0$$

where

$$g = \sigma^{-1}\left[\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \rho\right] \in \left(-\delta_{\mathbf{x}}, \lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)$$

$$c_0 > 0, \quad \mathbf{x}_0 = c_0 \mathbf{e}_1^T \left[\mathbf{I} - (g + \delta_{\mathbf{x}})\mathbf{A}\right]^{-1} > \mathbf{0}, \quad \mathbf{s}_0^T = \mathbf{x}_0^T \mathbf{A} > \mathbf{0}$$

and with supporting prices and rentals given by (49) where

$$\mathbf{v}_0 = c_0^{-\sigma}\mathbf{v}_{PF} > \mathbf{0}, \quad \mathbf{q}_0 = \left(\lambda_{PF}^{-1} - \delta\right)\mathbf{v}_0 > \mathbf{0}.$$

2. *If* $\lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1-\sigma) \leq \rho \leq a_{11}^{-1} - \delta_{\mathbf{x}}(1-\sigma)$ *then there is a cone of dimension* ≥ 1 *of price supported steady states classified as follows:* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ is a price supported steady state if and only if

   - *the growth rate is* $g = -\delta_{\mathbf{x}}$;
   - *it satisfies (29) with*

   $$c_0 > 0, \quad \mathbf{x}_0 = c_0 \mathbf{e}_1 + \boldsymbol{\alpha}, \quad \mathbf{s}_0^T = \mathbf{x}_0^T \mathbf{A} + \boldsymbol{\beta}$$

   where

   $$\boldsymbol{\alpha} \geq \mathbf{0}, \quad \boldsymbol{\alpha}^T \mathbf{v}_0 = \boldsymbol{\alpha}^T \mathbf{A} \mathbf{v}_0 = 0,$$
   $$\boldsymbol{\beta} \geq \mathbf{0}, \quad \boldsymbol{\beta}^T \mathbf{v}_0 = 0$$

   and $\mathbf{v}_0$ *is a solution of the system*

   $$\begin{cases} [\mathbf{I} - (\delta_{\mathbf{x}} + \rho - \sigma\delta_{\mathbf{x}}) \mathbf{A}] \mathbf{y} \leq \mathbf{0} \\ \mathbf{e}_1^T \mathbf{y} = c_0^{-\sigma} > 0 \\ \mathbf{e}_1^T [\mathbf{I} - (\delta_{\mathbf{x}} + \rho - \sigma\delta_{\mathbf{x}}) \mathbf{A}] \mathbf{y} = \mathbf{0} \\ \mathbf{y} \geq \mathbf{0}. \end{cases}$$

   In this case the steady state price-rental path $(\mathbf{v}, \mathbf{q})$ *is as in (49) and the stock path is given by:*

   $$\mathbf{s}_t^T = \left\{ \mathbf{s}_0^T - \frac{\boldsymbol{\alpha}}{\delta} - \boldsymbol{\beta} \right\} e^{-\delta_{\mathbf{x}} t} + \left\{ \frac{\boldsymbol{\alpha}}{\delta} + \boldsymbol{\beta} \right\} e^{-\delta_{\mathbf{z}} t}$$

   if $\delta \neq 0$, *and*

   $$\mathbf{s}_t^T = \left[\mathbf{s}_0^T + t\boldsymbol{\alpha} \right] e^{-\delta_{\mathbf{z}} t}$$

   if $\delta = 0$.

3. *If* $\rho > a_{11}^{-1} - \delta_{\mathbf{x}}(1-\sigma)$ *then there is a cone of dimension* n *of price supported steady states classified as follows:* $(\mathbf{s}_t, (\mathbf{x}_t, c_t))$ is a price supported steady state if and only if

   - *the growth rate is* $g = \sigma^{-1}\left[a_{11}^{-1} - \delta_{\mathbf{x}} - \rho\right] \in (-\infty, -\delta_{\mathbf{x}})$;
   - *it satisfies (29) with*

   $$c_0 > 0, \quad \mathbf{x}_0 = \frac{\sigma c_0}{\rho a_{11} - (1 - \delta_{\mathbf{x}} a_{11})(1-\sigma)} \mathbf{e}_1 + \boldsymbol{\alpha}, \quad \mathbf{s}_0 = \mathbf{x}_0^T \mathbf{A} + \boldsymbol{\beta}$$

   where

   $$\boldsymbol{\alpha} \geq \mathbf{0}, \quad \boldsymbol{\alpha}^T \mathbf{v}_0 = \boldsymbol{\alpha}^T \mathbf{A} \mathbf{v}_0 = 0,$$
   $$\boldsymbol{\beta} \geq \mathbf{0}, \quad \boldsymbol{\beta}^T \mathbf{v}_0 = 0$$

and $\mathbf{v}_0$ *is a solution of the system*

$$\begin{cases} \left[\mathbf{I}{-}a_{11}^{-1}\mathbf{A}\right]\mathbf{y} \leq \mathbf{0} \\ \mathbf{e}_1^T\mathbf{y} = c_0^{-\sigma} > 0 \\ \mathbf{e}_1^T\left[\mathbf{I}{-}a_{11}^{-1}\mathbf{A}\right]\mathbf{y} = \mathbf{0}; \\ \mathbf{y} \geq \mathbf{0}. \end{cases}$$

In this case the steady state price-rental path $(\mathbf{v}, \mathbf{q})$ *is as in (49) and the stock path is given by:*

$$\begin{aligned} \mathbf{s}_t^T &= \frac{e^{gt}}{g+\delta_{\mathbf{z}}}\left\{-\delta\mathbf{s}_0^T + \left[\left(\mathbf{x}_0^T\mathbf{e}_1 - c_0\right)\mathbf{e}_1^T + \boldsymbol{\alpha} + \delta\boldsymbol{\beta}\right]\right\} \\ &\quad + \frac{e^{-\delta_{\mathbf{z}}t}}{g+\delta_{\mathbf{z}}}\left\{(g+\delta_{\mathbf{x}})\,\mathbf{s}_0^T - \left[\left(\mathbf{x}_0^T\mathbf{e}_1 - c_0\right)\mathbf{e}_1^T + \boldsymbol{\alpha} + \delta\boldsymbol{\beta}\right]\right\}. \end{aligned}$$

if $g + \delta_{\mathbf{z}} \neq 0$ *and*

$$\mathbf{s}_t^T = e^{-\delta_{\mathbf{z}}t}\left\{\mathbf{s}_0^T + t\left[-\delta\mathbf{s}_0^T + \left(\mathbf{x}_0^T\mathbf{e}_1 - c_0\right)\mathbf{e}_1^T + \boldsymbol{\alpha} + \delta\boldsymbol{\beta}\right]\right\}$$

if $g + \delta_{\mathbf{z}} = 0$.

We state the following Lemmas which will be used in the proof of the main theorem.

Lemma 6.12 *In a price supported steady state the first process is always operated, i.e.*

$$\mathbf{x}_0^T\mathbf{e}_1 > 0.$$

Proof. Obtain from statement (ii) of Proposition 6.9 that $\mathbf{x} \neq \mathbf{0}$. Then assume on the contrary that $\mathbf{x}_0^T\mathbf{e}_1 = 0$. Then without loss of generality we can partition vectors $\mathbf{x}_0$, $\mathbf{v}_0$ and the matrix $\mathbf{A}$ as follows

$$\mathbf{x}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{bmatrix}; \qquad \mathbf{v}_0 = \begin{bmatrix} \mathbf{v}_{01} \\ \mathbf{v}_{02} \end{bmatrix}; \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

with $\mathbf{x}_2 > \mathbf{0}$, $\mathbf{v}_{02}$ has the same number of components as $\mathbf{x}_2$ and $\mathbf{A}_{22}$ is square and its order equals the number of components of $\mathbf{x}_2$.

Then by equations (48) and (49) we get

$$-\left(g+\delta_{\mathbf{x}}\right)\mathbf{x}_2^T\mathbf{A}_{21}\mathbf{v}_{01} = c_0\mathbf{e}_1^T\mathbf{v}_{01} \tag{55}$$

and

$$\mathbf{x}_2^T\left[\mathbf{I}{-}\left(g+\delta_{\mathbf{x}}\right)\mathbf{A}_{22}\right]\mathbf{v}_{02} = 0.$$

The former implies $g + \delta_{\mathbf{x}} < 0$ and $\mathbf{x}_2^T\mathbf{A}_{21}\mathbf{v}_{01} > 0$. Then the latter, implies $\mathbf{v}_{02} = \mathbf{0}$. Further, from equation (53) we get,

$$-\left(\rho+\delta_{\mathbf{x}}+\sigma g\right)\mathbf{x}_2^T\mathbf{A}_{21}\mathbf{v}_{01} = 0,$$

which contradicts (55) since (44) holds. ■

Lemma 6.13 *We have*

$$0 < g + \delta_{\mathbf{x}} < \lambda_{PF}^{-1} \Longleftrightarrow \mathbf{x}_0 > 0$$

and in this case

$$g = \sigma^{-1}\left(\lambda_{PF}^{-1} - \rho - \delta_{\mathbf{x}}\right); \qquad \mathbf{v}_0 = c_0^{-\sigma}\mathbf{v}_{PF} > 0. \tag{56}$$

Proof. In this proof we use the so called Frobenius Theorem for indecomposable matrices. If $0 < g + \delta_{\mathbf{x}} < \lambda_{PF}^{-1}$ then matrix $[\mathbf{I} - (g + \delta_{\mathbf{x}})\mathbf{A}]$ is invertible and $[\mathbf{I} - (g + \delta_{\mathbf{x}})\mathbf{A}]^{-1}$ is positive. Hence we obtain from (40) and Assumption 2.4 that $\mathbf{x}_0 > \mathbf{0}$.

If $\mathbf{x}_0 > \mathbf{0}$, then we get from (51), (52), (53) that

$$[\mathbf{I} - (\rho + \delta_{\mathbf{x}} + \sigma g)\mathbf{A}]\mathbf{v}_0 = \mathbf{0}, \qquad \mathbf{v}_0 \geq \mathbf{0}$$

which, taking account of equality (50), implies (56). Then, by equations (48) and (49) we obtain that

$$\mathbf{s}_0^T = \mathbf{x}_0^T\mathbf{A}, \qquad \mathbf{x}_0^T[\mathbf{I} - (g + \delta_{\mathbf{x}})\mathbf{A}] = c_0\mathbf{e}_1^T.$$

The last equation holds, with a positive $\mathbf{x}_0$, if and only if $0 < g + \delta_{\mathbf{x}} < \lambda_{PF}^{-1}$. ∎

Lemma 6.14 *Let* $\mathbf{A}$ *be a semipositive indecomposable matrix with* $a_{11} > 0$ *and let* H *be a given set of indices*

$$H = \{h_1, ..., h_z\}, \quad 1 < h_1 < ... < h_z.$$

Then the set

$$Z(\mathbf{A}, H) := \left\{\theta \in \mathbb{R} | \exists \mathbf{y} \in \mathbb{R}^n : \mathbf{y} \geq \mathbf{0}, \mathbf{e}_1^T\mathbf{y} > 0, [\mathbf{I} - \theta\mathbf{A}]\mathbf{y} \leq \mathbf{0}, \left[\mathbf{e}_1^T + \sum_{h \in H}\mathbf{e}_h^T\right][\mathbf{I} - \theta\mathbf{A}]\mathbf{y} = 0\right\}$$

is such that

$$\left[\lambda_{PF}^{-1}, a_{11}^{-1}\right] \supseteq Z(\mathbf{A}, H) \supseteq \left[\lambda_{PF}^{-1}, \lambda_1^{-1}\right)$$

where λ_1 *is the eigenvalue of maximum modulus of the matrix*

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{1h_1} & ... & a_{1h_z} \\ a_{h_1 1} & a_{h_1 h_1} & ... & a_{h_1 h_z} \\ \cdot & \cdot & ... & \cdot \\ \cdot & \cdot & ... & \cdot \\ a_{h_z 1} & a_{h_z h_1} & ... & a_{h_z h_z} \end{bmatrix}$$

Moreover, if H *is empty, then* $Z(A, H) = \left[\lambda_{PF}^{-1}, a_{11}^{-1}\right]$.

Proof. If $\theta < \lambda_{PF}^{-1}$, then there is no $\mathbf{y} \geq \mathbf{0}$ such that $[\mathbf{I} - \theta\mathbf{A}]\mathbf{y} \leq \mathbf{0}$ whereas for $\theta > a_{11}^{-1}$, there is no $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{e}_1^T\mathbf{y} > 0$, $\mathbf{e}_1^T[\mathbf{I} - \theta\mathbf{A}]\mathbf{y} = \mathbf{0}$. This proves the first inclusion. To prove the second inclusion observe first that, obviously, $\lambda_{PF}^{-1} \in Z(A, H)$.

Consider now the case of H nonempty. With no loss of generality assume that $H = \{2, 3, ..., z+1\}$ and partition $\mathbf{A}$ in four submatrices in such a way that matrix $\mathbf{B}$ is the submatrix $\mathbf{A}_{11}$ of $\mathbf{A}$. Then, if $\theta \in \left(\lambda_{PF}^{-1}, \lambda_1^{-1}\right)$, the vector

$$\mathbf{y} = \begin{pmatrix} \theta \left(\mathbf{I}-\theta \mathbf{A}_{11}\right)^{-1} \mathbf{A}_{12} \mathbf{v}_{PF2} \\ \mathbf{v}_{PF2} \end{pmatrix} \tag{57}$$

(where $\mathbf{v}_{PF2}$ denotes the last $n-(z+1)$ components of the Frobenius eigenvector of $\mathbf{A}$) satisfies the relevant (in)equalities since, by simple calculations one gets (since $\lambda_{PF}\theta < 1$)

$$\theta \left(\mathbf{I}-\theta \mathbf{A}_{11}\right)^{-1} \mathbf{A}_{12} \mathbf{v}_{PF2} > \mathbf{v}_{PF1}$$

(where $\mathbf{v}_{PF1}$ denotes the first $(z+1)$ components of the Frobenius eigenvector of $\mathbf{A}$).

Finally, if H is empty, then $\lambda_1 = a_{11}$ and the solution given in (57) works also in this case for $\theta \in \left(\lambda_{PF}^{-1}, a_{11}^{-1}\right)$. We have just to show that $a_{11}^{-1} \in Z(A, H)$, which is certainly the case since the vector $\mathbf{y} = \mathbf{e}_1$ satisfies the relevant (in)equalities. ■

Proof of Theorem 6.11. We show only the necessity part of our statements, i.e. that the steady states must satisfy cases 1-2-3. The sufficiency part, i.e. the fact that the solution proposed in the statement satisfies the steady state conditions, follows by straightforward calculations and we omit it.

Now, to prove the necessity we take any steady state and look at the steady state rate of growth. We have substantially three different possibilities (that form a complete partition of all possible price supported steady states):

(I) $g + \delta_{\mathbf{x}} > 0$.

(II) $g + \delta_{\mathbf{x}} = 0$.

(III) $g + \delta_{\mathbf{x}} < 0$.

We are going to prove that the first case corresponds to point 1 of the statement, the second case corresponds to point 2 and the third to point 3.

Case (I). Since Remark 6.8 holds, Lemma 6.13 holds. The boundaries for ρ mentioned in the claim follow from the constraints on g and equation (56)

Case (II) and (III). Since Lemma 6.12 applies we can assume, without loss of generality, that the positive components of $\mathbf{x}_0$ are the first $z+1$ ($z \in \{0, 1, ..., n-2\}$). Then we can write

$$\mathbf{x}_0 = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

with $\mathbf{x}_1 > \mathbf{0}$, $\mathbf{x}_1 \in \mathbb{R}^{z+1}$, $\mathbf{A}_{11}$ has order $z+1$. Since this part is quite long we divide it into several steps.

Step (i). *We claim that*

$$(g + \delta_{\mathbf{x}})\, \mathbf{x}_1^T \mathbf{A}_{12} \mathbf{v}_{02} = 0 \tag{58}$$

so that either $g + \delta_{\mathbf{x}} = 0$, *or* $\mathbf{A}_{12}\mathbf{v}_{02} = \mathbf{v}$, *or both, where* $(\mathbf{v}_{01}, \mathbf{v}_{02}) = \mathbf{v}_0$ *and* $\mathbf{v}_{01}$ *has the same number of components as* $\mathbf{x}_1$.

Inequalities (50) – (52) and equation (53) have a nonnegative solution if and only if

$$\rho + \delta_{\mathbf{x}} + \sigma g \in Z(\mathbf{A},H)$$

where H is the set of indices corresponding to the positive elements of $\mathbf{x}_0$ but index 1 and $Z(\mathbf{A},H)$ is as defined in Lemma 6.14. Since this lemma holds, we have $\rho + \delta_{\mathbf{x}} + \sigma g \in \left[\lambda_{PF}^{-1}, a_{11}^{-1}\right]$ which implies

$$g \in \left[\sigma^{-1}\left(\lambda_{PF}^{-1} - \rho - \delta_{\mathbf{x}}\right), \sigma^{-1}\left(a_{11}^{-1} - \rho - \delta_{\mathbf{x}}\right)\right]. \tag{59}$$

Equations (51) and (53) are equivalent to

$$\begin{aligned} \left[\mathbf{I}-(\rho + \delta_{\mathbf{x}} + \sigma g)\,\mathbf{A}_{11}\right]\mathbf{v}_{01} &= (\rho + \delta_{\mathbf{x}} + \sigma g)\,\mathbf{A}_{12}\mathbf{v}_{02} \\ \left[\mathbf{I}-(\rho + \delta_{\mathbf{x}} + \sigma g)\,\mathbf{A}_{22}\right]\mathbf{v}_{02} &\leq (\rho + \delta_{\mathbf{x}} + \sigma g)\,\mathbf{A}_{21}\mathbf{v}_{01}. \end{aligned} \tag{60}$$

Then, by admissibility (Proposition 6.7) we have

$$\mathbf{x}_0^T\left[\mathbf{I}-(g + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq c_0\mathbf{e}_1^T \quad if \quad g + \delta_{\mathbf{z}} \geq 0$$

$$\mathbf{x}_0^T(\mathbf{I} - \delta\mathbf{A}) - (g + \delta_{\mathbf{z}})\,\mathbf{s}_0^T \geq c_0\mathbf{e}_1^T \quad if \quad g + \delta_{\mathbf{z}} < 0$$

which means, by decomposing,

$$\mathbf{x}_1^T\left[\mathbf{I}-(g + \delta_{\mathbf{x}})\,\mathbf{A}_{11}\right] \geq c_0\mathbf{e}_1^T$$

$$-(g + \delta_{\mathbf{x}})\,\mathbf{x}_1^T\mathbf{A}_{12} \geq 0 \tag{61}$$

if $g + \delta_{\mathbf{z}} \geq 0$, and

$$\mathbf{x}_1^T\left[\mathbf{I}-\delta\mathbf{A}_{11}\right] - (g + \delta_{\mathbf{z}})\,\mathbf{s}_{01}^T \geq c_0\mathbf{e}_1^T$$

$$-\delta\mathbf{x}_1^T\mathbf{A}_{12} - (g + \delta_{\mathbf{z}})\,\mathbf{s}_{02}^T \geq 0$$

if $g+\delta_{\mathbf{z}} < 0$. Moreover the equality holds on the positive components of $\mathbf{v}_0$, (see Proposition 6.9-(ii), equation (48)) so that, for any sign of $g + \delta_{\mathbf{z}}$

$$\mathbf{x}_1^T\left[\mathbf{I}-(g + \delta_{\mathbf{x}})\,\mathbf{A}_{11}\right]\mathbf{v}_{01} = c_0^{1-\sigma} \tag{62}$$

$$-(g + \delta_{\mathbf{x}})\,\mathbf{x}_1^T\mathbf{A}_{12}\mathbf{v}_{02} = 0, \tag{63}$$

where we have also used the result of Proposition 6.7. Then we have the conclusion.

Step (ii). *Classification in the case* $g + \delta_{\mathbf{x}} = 0$.

Let then $g = -\delta_{\mathbf{x}}$. This implies that (thanks to (59))

$$\lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1 - \sigma) \leq \rho \leq a_{11}^{-1} - \delta_{\mathbf{x}}(1 - \sigma);$$

moreover, by (50) and (62) we get

$$\left(\mathbf{x}_1^T - c_0\mathbf{e}_1^T\right)\mathbf{v}_{01} = 0$$

so that $\mathbf{x}_0^T\mathbf{e}_1 = \mathbf{x}_1^T\mathbf{e}_1 = c_0$ and all the components of $\mathbf{v}_{01}$ but the first one are zero. The other complementarity conditions of the statement are consequences of (60) and (48)

Step (iii). *Classification in the case when* $\mathbf{A}_{12}\mathbf{v}_{02} = \mathbf{0}$ *and* $g + \delta_{\mathbf{x}} < 0$.

Now by the first of (60) we get

$$[\mathbf{I} - (\rho + \delta_{\mathbf{x}} + \sigma g) \mathbf{A}_{11}] \mathbf{v}_{01} = \mathbf{0}.$$

This means that $\mathbf{v}_{01}$ is an eigenvector of $\mathbf{A}_{11}$ with eigenvalue $(\rho + \delta_{\mathbf{x}} + \sigma g)^{-1}$. Let us call this eigenvalue λ_0 (obviously $\lambda_0 < \lambda_{PF}$). *Now let us show that*

$$\lambda_0 = a_{11},$$

that the other elements of the first column of $\mathbf{A}_{11}$ *(if* $\mathbf{A}_{11}$ *has order* > 1*) are zero and that, as a consequence,*

$$g_1 = \sigma^{-1} \left(a_{11}^{-1} - \rho - \delta_{\mathbf{x}} \right). \tag{64}$$

We have two possibilities:

- $\mathbf{v}_{01} > \mathbf{0}$. In this case we prove that $\mathbf{A}_{11}$ has order 1.

  Indeed by Proposition 6.9-(ii) we get that

  $$\mathbf{x}_1^T [\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{A}_{11}] = c_0 \mathbf{e}_1^T$$

  but this is impossible when $\mathbf{A}_{11}$ has order > 1, since $\mathbf{x}_1^T > 0$, $g + \delta_{\mathbf{x}} < 0$ and $c_0 \mathbf{e}_1^T$ has zero components.

- $\mathbf{v}_{01}$ has some zero components. In this case we prove that $\mathbf{A}_{11}$ is of the type

  $$\mathbf{A}_{11} = \begin{bmatrix} a_{11} & \mathbf{F}_{12} \\ \mathbf{0} & \mathbf{F}_{22} \end{bmatrix}$$

  where $\mathbf{F}_{12}$ and $\mathbf{F}_{22}$ are suitable nonnegative matrices.

  Indeed assume, without loss of generality, that the zero components of $\mathbf{v}_{01}$ are the last ones. Then we can write $\mathbf{x}_1 = (\mathbf{x}_{11}, \mathbf{x}_{12})$, $\mathbf{v}_{01} = (\mathbf{u}, \mathbf{0})$ with $\mathbf{u} > \mathbf{0}$, and

  $$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{0} & \mathbf{F}_{22} \end{bmatrix}.$$

  Moreover, taking account of the fact that $\mathbf{u} > \mathbf{0}$, we get, by Proposition 6.9-(ii)

  $$\mathbf{x}_{11}^T [\mathbf{I} - (g + \delta_{\mathbf{x}}) \mathbf{F}_{11}] = c_0 \mathbf{e}_1^T \tag{65}$$

  which implies, arguing as in the case $\mathbf{v}_{01} > 0$, that $\mathbf{F}_{11}$ has order 1, so the claim (64) holds.

Hence

$$\mathbf{A}\mathbf{v}_0 = \begin{bmatrix} a_{11} c_0^{-\sigma} \\ \mathbf{0} \\ \mathbf{A}_{21}\mathbf{v}_{01} + \mathbf{A}_{22}\mathbf{v}_{02} \end{bmatrix}, \qquad a_{11}\mathbf{x}_0^T \mathbf{v}_0 = \mathbf{x}_0^T \mathbf{A} \mathbf{v}_0 = c_0^{-\sigma}.$$

Being $g < -\delta_{\mathbf{x}}$ the above (64) is possible only if

$$\rho > a_{11}^{-1} - \delta_{\mathbf{x}} (1 - \sigma) .$$

Moreover, since the unique element of $\mathbf{x}_{11}$ is the scalar $\mathbf{x}_0^T \mathbf{e}_1$ we have by (65)

$$\mathbf{x}_0^T \mathbf{e}_1 [1 - (g + \delta_{\mathbf{x}}) a_{11}] = c_0$$

which gives

$$\mathbf{x}_0^T \mathbf{e}_1 = \frac{c_0}{1 - (g + \delta_{\mathbf{x}}) a_{11}} = \frac{\sigma c_0}{\rho a_{11} - (1 - \delta_{\mathbf{x}} a_{11}) (1 - \sigma)}.$$

The other complementarity conditions of the statement are consequences of (60) and (48)

■

6.5 Steady states and real rate of profits

Until now we never needed to deal with profitability and its rate(s). Since this subsection is devoted to investigating some macroeconomic insights connected with the model introduced in this paper, we need to provide these concepts. We deal first with concepts concerning profitability which can be introduced for all optimal paths. Then we turn to price supported steady state solutions Let $L_i(t_1, t_2)$ $(i = 1, 2, ..., n)$ be the amount of commodity i which can be obtained at time t_2 when one unit of commodity i is invested at time t_1 and let $L_0(t_1, t_2)$ be the amount of numeraire which can be obtained at time t_2 when one unit of numeraire is invested at time t_1. Obviously,

$$L_i(t_1, t_2) = \frac{L_0(t_1, t_2) \mathbf{e}_i^T \mathbf{v}_{t_1}}{\mathbf{e}_i^T \mathbf{v}_{t_2}} \qquad (i = 1, 2, ..., n)$$

$$L_j(t, t) = 1 \qquad \text{for each } t, j = 0, 1, ..., n$$

$$L_j(t_1, t) L_j(t, t_2) = L_j(t_1, t_2) \qquad \text{for each } t, j = 0, 1, ..., n$$

This allows us to define (for all t_1 and t_2)

$$r_j(t) := \frac{\frac{\partial L_j(t_1, t)}{\partial t}}{L_j(t_1, t)} = -\frac{\frac{\partial L_j(t, t_2)}{\partial t}}{L_j(t, t_2)} \qquad j = 0, 1, ..., n$$

The pure number $r_i(t)$ $(i = 1, 2, ..., n)$ is known in the literature as the "own rate of return of commodity i" (see, for instance, [23]), whereas $r_0(t)$ will be called here the "nominal rate of interest". Obviously

$$r_j(t) = r_0(t) - \frac{\mathbf{e}_j^T \dot{\mathbf{v}}_t}{\mathbf{e}_j^T \mathbf{v}_t} \tag{66}$$

which recalls the Fisher equation linking the nominal rate of interest, the real rate of interest, and the inflation rate. It is easily seen that

$$r_0(t) = \rho. \tag{67}$$

To prove this assume first that t_2 is such that in the range $[t, t_2]$ commodity h is produced ($\hat{\mathbf{x}}_\tau^T \mathbf{e}_h > 0$, each $\tau \in [t, t_2]$). Then $L_0(t, t_2)$ must satisfy the following equation (if the unit of numeraire is invested in the production of commodity h):

$$\int_t^{t_2} \frac{\mathbf{e}_h^T \mathbf{v}_\tau}{\mathbf{e}_h^T \mathbf{A} \mathbf{v}_t} L_0(\tau, t_2) e^{\delta_{\mathbf{x}}(t-\tau)} d\tau + e^{-\delta_{\mathbf{x}}(t_2 - t)} \frac{\mathbf{e}_h^T \mathbf{A} \mathbf{v}_{t_2}}{\mathbf{e}_h^T \mathbf{A} \mathbf{v}_t} = L_0(t, t_2)$$

that is,

$$\int_t^{t_2} \mathbf{e}_h^T \mathbf{v}_\tau L_0(\tau, t_2) e^{-\delta_{\mathbf{x}} \tau} d\tau + e^{-\delta_{\mathbf{x}} t_2} \mathbf{e}_h^T \mathbf{A} \mathbf{v}_{t_2} = L_0(t, t_2) \mathbf{e}_h^T \mathbf{A} \mathbf{v}_t e^{-\delta_{\mathbf{x}} t}$$

and differentiating with respect to t

$$-\mathbf{e}_h^T \mathbf{v}_t L_0(t, t_2) e^{-\delta_{\mathbf{x}} t} = \left[\mathbf{e}_h^T \mathbf{A} \dot{\mathbf{v}}_t - (r_0(t) + \delta_{\mathbf{x}})\, \mathbf{e}_h^T \mathbf{A} \mathbf{v}_t\right] L_0(t, t_2) e^{-\delta_{\mathbf{x}} t}.$$

Then the result is obtained by noting that in the range $[t, t_2]$

$$\mathbf{e}_h^T \mathbf{v}_t = (\rho + \delta_{\mathbf{x}})\, \mathbf{e}_h^T \mathbf{A} \mathbf{v}_t - \mathbf{e}_h^T \mathbf{A} \dot{\mathbf{v}}_t$$

since (19), (21), and (23) hold. Similarly, if in the range $[t, t_2]$ commodity h is not produced but it exists and is conserved ($\hat{\mathbf{x}}_\tau^T \mathbf{A} \mathbf{e}_h < \hat{\mathbf{s}}_\tau^T \mathbf{e}_h$, each $\tau \in [t, t_2]$). Then $L_0(t, t_2)$ must satisfy the following equation (if the unit of numeraire is invested in the conservation of commodity h):

$$e^{-\delta_{\mathbf{z}}(t_2 - t)} \frac{\mathbf{e}_h^T \mathbf{v}_{t_2}}{\mathbf{e}_h^T \mathbf{v}_t} = L_0(t, t_2),$$

that is,

$$e^{-\delta_{\mathbf{z}} t_2} \mathbf{e}_h^T \mathbf{v}_{t_2} = L_0(t, t_2) \mathbf{e}_h^T \mathbf{v}_t e^{-\delta_{\mathbf{z}} t},$$

and differentiating with respect to t

$$\left[\mathbf{e}_h^T \dot{\mathbf{v}}_t - (\delta_{\mathbf{z}} + r_0(t)) \mathbf{e}_h^T \mathbf{v}_t\right] L_0(t, t_2) \mathbf{e}_h^T \mathbf{v}_t e^{\delta_{\mathbf{z}} t} = 0.$$

Then the result is obtained by noting that in the range $[t, t_2]$

$$\mathbf{e}_h^T \dot{\mathbf{v}}_t = (\rho + \delta_{\mathbf{x}}) \mathbf{e}_h^T \mathbf{v}_t$$

since (19), (21), and (23) hold. From (66) and (67), taking account of the costate equation (19), we obtain

$$r_j(t) = \rho - \frac{\mathbf{e}_h^T \dot{\mathbf{v}}_t}{\mathbf{e}_h^T \mathbf{v}_t} = \frac{\mathbf{e}_h^T \mathbf{q}_t}{\mathbf{e}_h^T \mathbf{v}_t} - \delta_{\mathbf{z}}$$

which is the relationship between the price and the rental of an asset which if not used in production depreciates at rate $\delta_{\mathbf{z}}$. From it and equation (26) we obtain

$$\frac{\dot{c}_t}{c_t} = -\frac{1}{\sigma} \frac{\mathbf{e}_h^T \dot{\mathbf{v}}_t}{\mathbf{e}_h^T \mathbf{v}_t} = \frac{r_1(t) - \rho}{\sigma}$$

which can be also obtained from equations (20), (66) and (67).

By definition, in a steady state solution consumption c_t and the operation intensities of all processes $\mathbf{x}_t$ grow at a common rate g. Moreover, in a steady state solution supported by the steady state prices and rentals, all the own rates of return are equal since equations (66) hold. This common rate is also called the "real rate of profit", denoted by r. Hence in the steady state solutions mentioned in the Classification Theorem the relationship between the (real) rate of profit r and the growth rate g is the usual one contemplated in the endogenous growth literature:

$$g = \frac{r - \rho}{\sigma}.$$

The Classification Theorem can, in fact, be stated in terms of r rather than g:

- if $(\lambda_{PF}^{-1} - \delta_{\mathbf{x}})(1 - \sigma) < \rho < \lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1 - \sigma)$, then $r = \lambda_{PF}^{-1} - \delta_{\mathbf{x}}$;
- if $\lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1 - \sigma) \leq \rho \leq a_{11}^{-1} - \delta_{\mathbf{x}}(1 - \sigma)$, then $r = \rho - \sigma\delta_{\mathbf{x}}$;
- if $a_{11}^{-1} - \delta_{\mathbf{x}}(1 - \delta_{\mathbf{x}}) < \rho$, then $r = a_{11}^{-1} - \delta_{\mathbf{x}}$.

Let us define $\mathcal{R}$ and $\mathcal{G}$ as the sets of possible values of r and g, respectively, then $\mathcal{R} = [\lambda_{PF}^{-1} - \delta_{\mathbf{x}}, a_{11}^{-1} - \delta_{\mathbf{x}}]$, $\mathcal{G} = (-\infty, \lambda_{PF}^{-1} - \delta_{\mathbf{x}})$, and

$$Sup\mathcal{G} = min\mathcal{R}.$$

This relationship reminds one of the long period models studied in the sixties and seventies along the lines suggested by von Neumann and Sraffa ([34, pp.486-502] [20, pp.94-276] and the literature there referred to).

In the von Neumann model, for instance, the consumption is zero (the consumption by workers is included in the coefficients of matrix $\mathbf{A}$ as real wage rates) and the growth rate equals the profit rate. Hence the rate of profit is determined by technology alone (or, more precisely, by technology and real wage rates). That is exactly what happens in the model presented here if ρ, which however is a parameter concerning consumption preferences, is low enough or high enough. Only if $\lambda_{PF}^{-1} - \delta_{\mathbf{x}}(1 - \sigma) \leq \rho \leq a_{11}^{-1} - \delta_{\mathbf{x}}(1 - \sigma)$ does the rate of profit depend crucially on the parameters concerning consumption preferences; in this case, however, the rate of growth is determined by a technological parameter only. The reason is clear. If $g > -\delta_{\mathbf{x}}$, then all commodities need to be produced and therefore n processes are to be operated. Thus, the n equations relating prices and rate of profit relative to the operated processes (the no arbitrage conditions) determine both the n - 1 relative prices and the real rate of profit. If $g < -\delta_{\mathbf{x}}$, the only process which is relevant is the process producing commodity 1. Since the inputs used by this process are produced by the process itself at a rate larger than the growth rate (as a consequence of the assumption on depreciation), all these commodities (other than commodity 1) have a zero price. In other words, production is reduced to the production of commodity 1 by means of commodity 1 and free goods. Hence, similarly to the case of $g > -\delta_{\mathbf{x}}$, the equation relating prices and the rate of profit relative to the operated process of commodity 1 can determine the rate of profit (apart from commodity 1, all commodities which are either produced or conserved have a zero price). If $g = -\delta_{\mathbf{x}}$, then once again the only relevant process is that producing commodity 1 and the inputs used by this process are produced by the process itself, but this is realized at a rate equal to the growth rate and therefore these commodities (except commodity 1) may have either a positive or a zero price. Those with a positive price cannot be separately produced or conserved; their existing stocks can be regarded as stocks of 'renewable' resources for which a growth rate of $-\delta_{\mathbf{x}}$ can be granted in the production of commodity 1.

For optimal steady states with a growth rate larger than $-\delta_{\mathbf{x}}$ the rate of growth is determined as in the simple one-sector AK model with the von Neumann rate of profit replacing the coefficient A. Thus, to study these paths 'no fancy mathematics from the calculus of variations is needed, much less an appeal to Pontryagin' ([22, p.11], see also [30]). We note, however, that the existence of such a structure in the set of the steady states is simply the way in which the Non-substitution Theorem - introduced in 1951 by Arrow, Georgescu-Roegen, Koopmans, and Samuelson) (see [20, pp.26-8, 151-2, 270] and the literature there referred to) - operates in the present endogenous growth framework. In fact the Non-substitution Theorem informs us that in the interior of the production set, even if there is more than one process for each commodity to be produced, the long run supply curves are horizontal at the prices (and the rate of profit) given by the dominant technique (that is the set of n processes, one for each commodity to be produced, which can pay the largest rate of profit). This means that the preference side of the model can affect prices at the boundary of the production set, but if all processes are positively operated, then the long run equilibrium prices (and the rate of profit) are determined from the production side of the model only. In this case the role of preferences (including ρ), is to determine the growth rate (and, when more than one commodity can be consumed, the proportions in which commodities are consumed). In the present model the Non-substitution Theorem implies that the rate of profit and prices are independent of the growth rate, provided it is larger than $-\delta_{\mathbf{x}}$. However, when it has been established that the case in which the growth rate is lower than $-\delta_{\mathbf{x}}$ is equivalent to the case in which commodity 1 is produced by itself (and free goods) alone, then the Non-substitution Theorem can be invoked again to assert that the rate of profit is independent of the growth rate, also when the growth rate is lower than, but not equal to $-\delta_{\mathbf{x}}$.

The cases of optimal steady states with a growth rate lower than or equal to $-\delta_{\mathbf{x}}$ are a consequence of the assumption on depreciation. Without a depreciation by evaporation they could not exist. However, their relevance is not to be underestimated. It is interesting that with a high rate of time preference the consumer exploits the stocks of 'renewable' resources and with an even higher rate the consumer exploits not only these stocks, but the capital stocks themselves. (If the reader allows us to use this model for an interpretation of real phenomena, we can say that in a country in which the political leadership is highly dubious about its ability to preserve its rule and, as a consequence, has a high rate of time preference, it will concentrate production in exploiting the stocks of natural resources and the growth rate will be very low or even negative, whereas in a country in which the political leadership has an even higher rate of time preference, it will concentrate production not only in exploiting the stocks of natural resources but also in exploiting the previously accumulated capital and the growth rate will be lower and certainly negative.)

A Proof of the existence theorem

We now prove the above Theorem 3.1 about existence of optimal strategies. The proof uses quite straightforward compactness arguments but they are combined in a non-standard way and, to our knowledge, the results given in the literature do not apply to this case (see [11] and [32] for similar results). For this reason we give a complete proof. Troughout this subsection we will assume that Assumptions 2.2 and 2.3 hold true without mentioning them. We will clarify when other assumptions are used.

We start giving some preliminary results that will be useful in the following.

Lemma A.1 *The set of admissible control strategies is convex and the functional U_σ is strictly concave with respect to the argument c.*

Proof. It is immediate and we omit it: see Appendix B for related results (Lemma B.1 and Proposition B.2). ∎

Now we prove the following useful estimates (see [15, p.30] for analogous arguments in the one-dimensional case). Recall that $\mathbf{v}_{PF}$ is the Frobenius right eigenvector of $\mathbf{A}$ and so $\mathbf{v}_{PF} > \mathbf{0}$.

Lemma A.2 *Let $\sigma > 0$. For every $0 \le t \le \tau < +\infty$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \ge 0$ we have, for every admissible control strategy $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$,*

$$\mathbf{s}_\tau^T \mathbf{v}_{PF} \le e^{(\lambda_{PF} - 1 - \delta_{\mathbf{x}})(\tau - t)} \mathbf{s}_t^T \mathbf{v}_{PF}, \tag{68}$$

and, for $\eta \in \mathbb{R}$

$$\begin{aligned}\int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds &\le \mathbf{s}_t^T \mathbf{v}_{PF} e^{-\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} \frac{e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)\tau} - e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)t}}{\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta}; \quad \eta \ne \lambda_{PF}^{-1} - \delta_{\mathbf{x}} \\ \int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds &\le \mathbf{s}_t^T \mathbf{v}_{PF} e^{-\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} (\tau - t); \quad \eta = \lambda_{PF}^{-1} - \delta_{\mathbf{x}}\end{aligned} \tag{69}$$

and, setting $I(t, \tau) := \int_t^\tau c_s ds$,

$$I(t, \tau) + \lambda_{PF} \left(\mathbf{x}_\tau^T \mathbf{v}_{PF}\right) \le e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(\tau - t)} \mathbf{s}_t^T \mathbf{v}_{PF}, \tag{70}$$

and also

$$\lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} e^{-\eta\tau} + \int_t^\tau e^{-\eta s} c_s ds \le e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF} e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)^+ (\tau - t)} \tag{71}$$

Moreover, setting $a = \rho - \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1 - \sigma)$ we have for $\sigma \in (0, 1)$

$$\begin{aligned}&\int_t^\tau e^{-\rho s} c_s^{1-\sigma} ds \\ \le\ & e^{-\rho t} [\mathbf{s}_t^T \mathbf{v}_{PF}]^{1-\sigma} \left[(\tau - t)^\sigma e^{-a(\tau - t)} + [\rho]^+ \int_t^\tau (s - t)^\sigma e^{-a(s - t)} ds\right]\end{aligned} \tag{72}$$

while, for $\sigma \in (1,+\infty)$

$$\begin{aligned} & \int_t^\tau e^{-\rho s} c_s^{1-\sigma} ds \qquad (73) \\ \geq\ & (\tau-t)^\sigma e^{-\rho t} \left(\mathbf{s}_t^T \mathbf{v}_{PF}\right)^{1-\sigma} e^{(1-\sigma)\left(\frac{a}{\sigma-1}\right)^+ (\tau-t)} \end{aligned}$$

and, for $\sigma = 1$

$$\begin{aligned} \int_t^\tau e^{-\rho s} \log c_s ds \ \leq\ & e^{-\rho\tau} (\tau-t) \left[\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(\tau-t) + \log \frac{\mathbf{s}_t^T \mathbf{v}_{PF}}{(\tau-t)}\right] \qquad (74) \\ & + [\rho]^+ \int_t^\tau e^{-\rho s} (s-t) \left[\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(s-t) + \log \frac{\mathbf{s}_t^T \mathbf{v}_{PF}}{(s-t)}\right] ds \end{aligned}$$

Proof. We prove the seven inequalities (68)–(74) in order of presentation.

1. First we observe that, by multiplying the state equation (4) by $\mathbf{v}_{PF}$ we obtain

$$\begin{aligned} \dot{\mathbf{s}}_t^T \mathbf{v}_{PF} &= -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_{PF} + (1-\delta\lambda_{PF}) \mathbf{x}_t^T \mathbf{v}_{PF} - c_t \quad t \in (0,+\infty), \qquad (75) \\ \mathbf{s}_0^T \mathbf{v}_{PF} &= \bar{\mathbf{s}}^T \mathbf{v}_{PF} \geq 0 \end{aligned}$$

   Moreover, the constraint $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ becomes $\mathbf{s}_t^T \mathbf{v}_{PF} \geq \lambda_{PF} \mathbf{x}_t^T \mathbf{v}_{PF}$ so that, from (75), from the fact that $1-\delta\lambda_{PF} > 0$ (consequence of Assumption 2.3), and from the nonnegativity of c, we get

$$\dot{\mathbf{s}}_s^T \mathbf{v}_{PF} \leq \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \mathbf{s}_s^T \mathbf{v}_{PF} - c_s \leq \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \mathbf{s}_s^T \mathbf{v}_{PF} \qquad s \in (0,+\infty), \qquad (76)$$

   and so, by integrating on $[t,\tau]$ and using the Gronwall lemma (see e.g. [3, p. 218]) we get the first claim (68).

2. To prove inequality (69) we multiply the inequality (68) by $e^{-\eta s}$ and integrate. We obtain for $\eta \neq \lambda_{PF}^{-1} - \delta_{\mathbf{x}}$

$$\begin{aligned} \int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds &\leq \int_t^\tau e^{-\eta s} e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(s-t)} \mathbf{s}_t^T \mathbf{v}_{PF} ds \\ &= \mathbf{s}_t^T \mathbf{v}_{PF} e^{-\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} \int_t^\tau e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)s} ds \\ &= \mathbf{s}_t^T \mathbf{v}_{PF} e^{-\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} \frac{e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)\tau} - e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)t}}{\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta} \end{aligned}$$

   whereas if $\eta = \lambda_{PF}^{-1} - \delta_{\mathbf{x}}$ we have

$$\begin{aligned} \int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds &\leq \int_t^\tau e^{-\eta s} e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(s-t)} \mathbf{s}_t^T \mathbf{v}_{PF} ds \\ &= \mathbf{s}_t^T \mathbf{v}_{PF} e^{-\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} (\tau-t). \end{aligned}$$

3. For the third claim (70) we observe that, from (76)

$$0 \le c_s \le \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \mathbf{s}_s^T \mathbf{v}_{PF} - \dot{\mathbf{s}}_s^T \mathbf{v}_{PF} \qquad \forall s \in [t, \tau] \tag{77}$$

so that, by integrating on $[t, \tau]$

$$0 \le I(t, \tau) = \int_t^\tau c_s ds \le \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \int_t^\tau \mathbf{s}_s^T \mathbf{v}_{PF} ds - \mathbf{s}_\tau^T \mathbf{v}_{PF} + \mathbf{s}_t^T \mathbf{v}_{PF}$$

and from the inequalities (68) and $\lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} \le \mathbf{s}_\tau^T \mathbf{v}_{PF}$

$$\begin{aligned} I(t, \tau) &\le \int_t^\tau \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) e^{\left(1/\lambda_{PF} - \delta_{\mathbf{x}}\right)(s-t)} \mathbf{s}_t^T \mathbf{v}_{PF} ds - \lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} + \mathbf{s}_t^T \mathbf{v}_{PF} \\ &= e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(\tau - t)} \mathbf{s}_t^T \mathbf{v}_{PF} - \lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} \end{aligned}$$

which gives the third claim (70).

4. The fourth claim (71) easily follows by multiplying both sides of (77) by $e^{-\eta s}$ and then integrating. In fact we have

$$0 \le e^{-\eta s} c_s \le e^{-\eta s} \left[\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \mathbf{s}_s^T \mathbf{v}_{PF} - \dot{\mathbf{s}}_s^T \mathbf{v}_{PF}\right] \qquad \forall s \in [t, \tau]$$

and integrating and using that $\lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} \le \mathbf{s}_\tau^T \mathbf{v}_{PF}$

$$\begin{aligned} \int_t^\tau e^{-\eta s} c_s ds &\le \int_t^\tau e^{-\eta s} \left[\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \mathbf{s}_s^T \mathbf{v}_{PF} - \dot{\mathbf{s}}_s^T \mathbf{v}_{PF}\right] ds \\ &= \int_t^\tau e^{-\eta s} \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) \mathbf{s}_s^T \mathbf{v}_{PF} ds \\ &\quad - e^{-\eta \tau} \mathbf{s}_\tau^T \mathbf{v}_{PF} + e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF} \\ &\quad - \eta \int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds \\ &\le e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF} - e^{-\eta \tau} \lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} \\ &\quad + \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right) \int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds \end{aligned}$$

Now, if $\eta \ge \lambda_{PF}^{-1} - \delta_{\mathbf{x}}$ the above inequality implies

$$\int_t^\tau e^{-\eta s} c_s ds + e^{-\eta \tau} \lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} \le e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF}$$

while, for $\eta < \lambda_{PF}^{-1} - \delta_{\mathbf{x}}$ we get, by using (69),

$$\int_t^\tau e^{-\eta s} c_s ds + e^{-\eta \tau} \lambda_{PF} \mathbf{x}_\tau^T \mathbf{v}_{PF} \le e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF} + \left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right) \int_t^\tau e^{-\eta s} \mathbf{s}_s^T \mathbf{v}_{PF} ds$$

$$\begin{aligned} &\le e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF} + \mathbf{s}_t^T \mathbf{v}_{PF} e^{-\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} \left[e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)\tau} - e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)t}\right] \\ &= e^{-\eta t} \mathbf{s}_t^T \mathbf{v}_{PF} e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \eta\right)(\tau - t)} \end{aligned}$$

which gives the fourth claim (71)

5. Concerning the fifth inequality (72) setting (see e.g. [15, p. 30])

$$h(t,s) = \int_t^s c_r^{1-\sigma} dr$$

we have, by Jensen's inequality, for $\sigma \in (0,1)$

$$h(t,s) \le (s-t)\left[\frac{1}{s-t}\int_t^s c_r dr\right]^{1-\sigma} = (s-t)^\sigma I(t,s)^{1-\sigma} \tag{78}$$

Now, integrating by parts we obtain (this holds in fact for $\sigma > 0$, $\sigma \neq 1$),

$$\begin{aligned}\int_t^\tau e^{-\rho s} c_r^{1-\sigma} ds &= \left[e^{-\rho s} h(t,s)\right]_t^\tau + \int_t^\tau \rho e^{-\rho s} h(t,s) ds \\ &= e^{-\rho\tau} h(t,\tau) + \int_t^\tau \rho e^{-\rho s} h(t,s) ds.\end{aligned} \tag{79}$$

If we apply the inequality (70) to (78) we obtain

$$h(t,s) \le (s-t)^\sigma e^{(1-\sigma)(\lambda_{PF}^{-1}-\delta_{\mathbf{x}})(s-t)} [\mathbf{s}_t^T \mathbf{v}_{PF}]^{1-\sigma}$$

which yields, together with (79),

$$\int_t^\tau e^{-\rho s} c_s^{1-\sigma} ds$$

$$\begin{aligned}\le\ & e^{-\rho\tau}(\tau-t)^\sigma e^{(1-\sigma)(\lambda_{PF}^{-1}-\delta_{\mathbf{x}})(\tau-t)} [\mathbf{s}_t^T \mathbf{v}_{PF}]^{1-\sigma} \\ & + [\rho]^+ \int_t^\tau e^{-\rho s}(s-t)^\sigma e^{(1-\sigma)(\lambda_{PF}^{-1}-\delta_{\mathbf{x}})(s-t)} [\mathbf{s}_t^T \mathbf{v}_{PF}]^{1-\sigma} ds\end{aligned}$$

$$= e^{-\rho t}(\tau-t)^\sigma e^{-a(\tau-t)} [\mathbf{s}_t^T \mathbf{v}_{PF}]^{1-\sigma} + [\rho]^+ e^{-\rho t} [\mathbf{s}_t^T \mathbf{v}_{PF}]^{1-\sigma} \int_t^\tau (s-t)^\sigma e^{-a(s-t)} ds$$

which gives the claim.

6. To prove inequality (73) for the case when $\sigma \in (1,+\infty)$ we apply directly the Jensen inequality to the integral $\int_t^\tau e^{-\rho s} c_s^{1-\sigma} ds$. In fact

$$\begin{aligned}\int_t^\tau e^{-\rho s} c_s^{1-\sigma} ds &= (\tau-t)\frac{1}{\tau-t}\int_t^\tau \left(e^{-\frac{\rho}{1-\sigma}s} c_s\right)^{1-\sigma} ds \\ &\ge (\tau-t)\left[\frac{1}{\tau-t}\int_t^\tau e^{-\frac{\rho}{1-\sigma}s} c_s ds\right]^{1-\sigma} \\ &= (\tau-t)^\sigma \left[\int_t^\tau e^{-\frac{\rho}{1-\sigma}s} c_s ds\right]^{1-\sigma}\end{aligned}$$

so that, by inequality (71) with $\eta = \frac{\rho}{1-\sigma}$ we get , (recalling that $\lambda_{PF}^{-1} - \delta_{\mathbf{x}} - \frac{\rho}{1-\sigma} = \frac{a}{\sigma-1}$)

$$\begin{aligned}\int_t^\tau e^{-\rho s} c_s^{1-\sigma} ds &\geq (\tau - t)^\sigma \left[\int_t^\tau e^{-\frac{\rho}{1-\sigma}s} c_s ds\right]^{1-\sigma} \\ &\geq (\tau - t)^\sigma \left[e^{-\frac{\rho}{1-\sigma}t} \mathbf{s}_t^T \mathbf{v}_{PF} e^{\left(\frac{a}{\sigma-1}\right)^+(\tau-t)}\right]^{1-\sigma} \\ &= (\tau - t)^\sigma e^{-\rho t} \left(\mathbf{s}_t^T \mathbf{v}_{PF}\right)^{1-\sigma} e^{(1-\sigma)\left(\frac{a}{\sigma-1}\right)^+(\tau-t)}\end{aligned}$$

Note that for the case $\sigma \in (0, 1)$ we are interested in an estimate from above giving finiteness of the value function for $a > 0$ (so we need terms that remain bounded when $t \to +\infty$), while for the case $\sigma \in (1, \infty)$ we are interested in an estimate from below giving the value function equal to $-\infty$ when $a \leq 0$, (so we need terms that explode when $t \to +\infty$). These different targets require to use different estimates with different methods of proof. Of course, both methods can be applied to both cases yielding however estimates that are not useful for our target.

7. Inequality (74) follows by similar arguments. In fact, calling

$$h(t, s) = \int_t^s \log c_r dr$$

we have, because of Jensen's inequality

$$h(t, s) \leq (s - t) \log \left[\frac{1}{s - t} \int_t^s c_r dr\right] = (s - t) \left[-\log (s - t) + \log I(t, s)\right]. \quad (80)$$

Now, integrating by parts as in (79), we obtain

$$\int_t^\tau e^{-\rho s} \log c_s ds = e^{-\rho\tau} h(t, \tau) + \int_t^\tau \rho e^{-\rho s} h(t, s) ds. \quad (81)$$

which, together with (80) and (70), gives

$$\begin{aligned}&\int_t^\tau e^{-\rho s} \log c_s ds \\ \leq\ & e^{-\rho\tau}(\tau - t) \log \left[\frac{1}{\tau - t} I(t, \tau)\right] + \int_t^\tau [\rho]^+ e^{-\rho s}(s - t) \log \left[\frac{1}{s - t} I(t, s)\right] ds \\ \leq\ & e^{-\rho\tau}(\tau - t) \log \left[\frac{1}{\tau - t} e^{(\lambda_{PF}^{-1} - \delta_{\mathbf{x}})(\tau - t)} \mathbf{s}_t^T \mathbf{v}_{PF}\right] \\ & + [\rho]^+ \int_t^\tau e^{-\rho s}(s - t) \log \left[\frac{1}{s - t} e^{(\lambda_{PF}^{-1} - \delta_{\mathbf{x}})(s - t)} \mathbf{s}_t^T \mathbf{v}_{PF}\right] ds \\ =\ & e^{-\rho\tau}(\tau - t) \left[(\lambda_{PF}^{-1} - \delta_{\mathbf{x}})(\tau - t) + \log \frac{\mathbf{s}_t^T \mathbf{v}_{PF}}{\tau - t}\right] \\ & + [\rho]^+ \int_t^\tau e^{-\rho s}(s - t) \left[(\lambda_{PF}^{-1} - \delta_{\mathbf{x}})(s - t) + \log \frac{\mathbf{s}_t^T \mathbf{v}_{PF}}{s - t}\right] ds\end{aligned}$$

which completes the proof. ∎

Remark A.3 We observe that, due to the strict positivity of $\mathbf{v}_{PF}$ (which comes from indecomposability of $\mathbf{A}$), the above estimates are in fact estimates on every component of the vectors $\mathbf{s}_t$, $\mathbf{x}_t$. This will be useful in the rest of the section. In particular from the above Lemma A.1 we can derive conditions for the functional U_σ to be well defined and finite for every admissible strategy when $\sigma \in (0,1]$ and Assumption 2.7 holds, i.e. $a > 0$, and for U_σ to be $-\infty$ for every admissible strategy when $\sigma \in (1,+\infty)$ and $a \leq 0$. ∎

The following corollary is immediate (using estimates (72) (73), (74) and, when $a > 0$, that $\int_0^{+\infty} s^\sigma e^{-as}\, ds = \frac{\Gamma(1+\sigma)}{a^{1+\sigma}}$) and gives a first part of Theorem 3.1.

Corollary A.4 *Let Assumption 2.7 hold, i.e. $a > 0$. Then, for any $\overline{\mathbf{s}} \geq \mathbf{0}$ we have, for $\sigma \in (0,1)$ and $(\mathbf{x}, c) \in \mathcal{A}(\overline{\mathbf{s}})$,*

$$\begin{aligned} 0 \ &\leq\ U_\sigma(c) \leq \frac{\rho}{1-\sigma}\left[\overline{\mathbf{s}}^T \mathbf{v}_{PF}\right]^{1-\sigma} \int_0^{+\infty} s^\sigma e^{-as}\, ds \\ &=\ \frac{\rho}{1-\sigma}\frac{\Gamma(1+\sigma)}{a^{1+\sigma}}\left[\overline{\mathbf{s}}^T \mathbf{v}_{PF}\right]^{1-\sigma} \end{aligned}$$

while, for $\sigma = 1$

$$U_\sigma(c) \leq \rho \int_0^{+\infty} e^{-\rho s} s \left[\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right) s + \log \frac{\overline{\mathbf{s}}^T \mathbf{v}_{PF}}{s}\right] ds$$

and, for $\sigma > 1$

$$U_\sigma(c) \leq 0.$$

Moreover, if $a \leq 0$, then for $\sigma > 1$, and for $\sigma = 1$ and $\lambda_{PF}^{-1} - \delta_{\mathbf{x}} = 0$ every admissible strategy satisfies $U_\sigma(c) = -\infty$.

This result show, in particular, that, when $a > 0$ and $\sigma \in (0,1)$, the intertemporal utility functional $U_\sigma(c)$ is finite and uniformly bounded for every admissible production-consumption strategy (while for $\sigma \geq 1$ it is only bounded from above). In the case when $a \leq 0$, $\sigma > 1$, there are no optimal strategies.

We now have the following result.

Corollary A.5 *Let Assumption 2.7 do not hold., i.e. $a \leq 0$. Let either $\sigma \in (0,1)$ or $\sigma = 1$ and $\lambda_{PF}^{-1} - \delta_{\mathbf{x}} > 0$. Then, given any $\overline{\mathbf{s}} > \mathbf{0}$ we can find an admissible strategy $(\mathbf{x}, c) \in \mathcal{A}(\overline{\mathbf{s}})$ such that $U_\sigma(c) = +\infty$. If on the other hand $a > 0$ and $\sigma \geq 1$, then there exists an admissible strategy with $U_\sigma(c) > -\infty$.*

Proof. Let first $a \leq 0$ and $\sigma \in (0,1]$. Take a positive decreasing function $\alpha : [0,+\infty) \longmapsto [0,+\infty)$ defined as $\alpha(t) = \alpha_0 (1+t)^{-\gamma}$ for suitable $\alpha_0 > 0$ and $\gamma > 0$ (to be fixed later on). Set $\mathbf{x}_t = \alpha(t) e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t} \mathbf{s}_{PF}$ and $c_t = -\lambda_{PF}\alpha'(t) e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)t}$ (recall that $\mathbf{s}_{PF}$ is the left positive eigenvector of $\mathbf{A}$ with the first component equal to 1). Then the associated solution of the state equation (4) is given by:

$$\mathbf{s}_t^T = e^{-\delta_{\mathbf{z}} t}\bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_s^T \left[\mathbf{I} - \delta \mathbf{A}\right] ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} c_s \mathbf{e}_1^T ds$$

$$= e^{-\delta_{\mathbf{z}} t}\left[\bar{\mathbf{s}}^T + \mathbf{s}_{PF}^T (1 - \delta\lambda_{PF}) \int_0^t \alpha(s) e^{\left(\lambda_{PF}^{-1} - \delta\right)s} ds + \mathbf{e}_1^T \int_0^t \lambda_{PF}\alpha'(s) e^{\left(\lambda_{PF}^{-1} - \delta\right)s} ds\right]$$

Integrating by parts the first integral becomes:

$$\begin{aligned} & (1 - \delta\lambda_{PF}) \int_0^t \alpha(s) e^{\left(\lambda_{PF}^{-1} - \delta\right)s} ds \\ = \; & \lambda_{PF}\left[\alpha(t) e^{\left(\lambda_{PF}^{-1} - \delta\right)t} - \alpha_0 - \int_0^t \alpha'(s) e^{\left(\lambda_{PF}^{-1} - \delta\right)s} ds\right] \end{aligned}$$

so that

$$\begin{aligned} \mathbf{s}_t^T \; = \; & e^{-\delta_{\mathbf{z}} t}\bar{\mathbf{s}}^T + \lambda_{PF}\mathbf{s}_{PF}^T \alpha(t) e^{(1/\lambda_{PF} - \delta_{\mathbf{x}})t} - \lambda_{PF}\mathbf{s}_{PF}^T \alpha_0 e^{-\delta_{\mathbf{z}} t} \\ & + e^{-\delta_{\mathbf{z}} t}\lambda_{PF}\left[-\mathbf{s}_{PF}^T + \mathbf{e}_1^T\right] \int_0^t \alpha'(s) e^{(1/\lambda_{PF} - \delta)s} ds \end{aligned}$$

$$= \mathbf{x}_t^T \mathbf{A} + e^{-\delta_{\mathbf{z}} t}\left[\bar{\mathbf{s}}^T - \alpha_0 \lambda_{PF}\mathbf{s}_{PF}^T\right] + \lambda_{PF} e^{-\delta_{\mathbf{z}} t}\left[-\mathbf{s}_{PF}^T + \mathbf{e}_1^T\right] \int_0^t \alpha'(s) e^{(1/\lambda_{PF} - \delta_{\mathbf{x}})s} ds$$

It is clear that the constraints $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ are satisfied if

$$\bar{\mathbf{s}}^T - \alpha_0 \lambda_{PF}\mathbf{s}_{PF}^T \geq 0;$$

which is always possible by taking α_0 sufficiently small (since $\bar{\mathbf{s}} > 0$).

Now we observe that the above strategy is admissible but we have, for $\sigma \in (0,1)$,

$$U_\sigma(c) = \frac{1}{1-\sigma}\int_0^{+\infty} e^{-\rho s} c_s^{1-\sigma} ds = \frac{\lambda_{PF}^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{-\rho s} (-\alpha'(s))^{1-\sigma} e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)(1-\sigma)s} ds$$

$$= \frac{\lambda_{PF}^{1-\sigma}(\gamma\alpha_0)^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{-as}(1+s)^{(-1-\gamma)(1-\sigma)} ds \geq \frac{(\gamma\alpha_0)^{1-\sigma}}{1-\sigma}\int_0^{+\infty} (1+s)^{(-1-\gamma)(1-\sigma)} ds$$

and the last integral is infinite if $\gamma < \frac{\sigma}{1-\sigma}$). Moreover, for $\sigma = 1$

$$U_\sigma(c) = \int_0^{+\infty} e^{-\rho s} \log c_s ds = \int_0^{+\infty} e^{-\rho s}\left(\log \lambda_{PF}\left(-\alpha'(s)\right) e^{\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)s}\right) ds.$$

If $\lambda_{PF}^{-1}-\delta_{\mathbf{x}}>0$ the function $e^{-\rho s}\log\lambda_{PF}\left(-\alpha'\left(s\right)\right)e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)s}$ is locally bounded, definitely positive, and goes to $+\infty$ for $s\to+\infty$. Then $U_\sigma(c)=+\infty$.

Let $a>0$ and $\sigma\in[1,+\infty)$. We observe that the strategy $\mathbf{x}_t=\alpha(t)e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)t}\mathbf{s}_{PF}$ and $c_t=-\lambda_{PF}\alpha'(t)e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)t}$ with $\alpha(t)=\alpha_0\left(1+t\right)^{-\gamma}$, $\gamma>0$, $\bar{\mathbf{s}}^T-\alpha_0\lambda_{PF}\mathbf{s}_{PF}^T\geq 0$, is still admissible (since admissibility does no depend on the value of σ. We then have, for $\sigma\in(1,+\infty)$

$$U_\sigma(c)=\frac{1}{1-\sigma}\int_0^{+\infty}e^{-\rho s}c_s^{1-\sigma}ds=\frac{\lambda_{PF}^{1-\sigma}}{1-\sigma}\int_0^{+\infty}e^{-\rho s}(-\alpha'(s))^{1-\sigma}e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)(1-\sigma)s}ds$$

$$=\frac{\lambda_{PF}^{1-\sigma}(\gamma\alpha_0)^{1-\sigma}}{1-\sigma}\int_0^{+\infty}e^{-as}(1+s)^{(-1-\gamma)(1-\sigma)}ds.$$

Since $a>0$ the integral is finite irrespective of the value of $\gamma>0$. For $\sigma=1$ we have

$$U_\sigma(c)=\int_0^{+\infty}e^{-\rho s}\log c_s ds=\int_0^{+\infty}e^{-\rho s}\left(\log\left(\lambda_{PF}\left(-\alpha'\left(s\right)\right)e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)s}\right)\right)ds$$

$$=\int_0^{+\infty}e^{-\rho s}\left(\log\left(\lambda_{PF}\alpha_0\gamma(1+s)^{-1-\gamma}e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)s}\right)\right)ds.$$

Since the function $\log\left[\lambda_{PF}\alpha_0\gamma(1+s)^{-1-\gamma}e^{\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)s}\right]$ is less than polinomially growing and $\rho=a>0$ then the integral above is finite, so $U_\sigma(c)>-\infty$. ∎

Remark A.6 We observe that, in the case when $a\leq 0$, even if the supremum of the intertemporal utility U_σ is infinite, the optimal control problem could be studied in a similar way by using more general concepts of optimality for infinite horizon control problem (e.g the overtaking optimality: see [32, pp. 231-233]). We will not do it here for simplicity. So we will always assume $a>0$. ■

We now prove the existence (and uniqueness) result for optimal strategies when $a>0$. We use compactness for weak topologies, see for a reference on this e.g. [8] or [35].

Lemma A.7 *Assume that $a>0$. Then there exists an optimal production-consumption strategy $(\mathbf{x},c)$ maximizing U_σ. This strategy is unique in the sense that, if $(\hat{\mathbf{x}},\hat{c})$ is another optimal strategy, then $\hat{c}=c$ a.e..*

Proof. We give the proof for the case $\sigma\in(0,1)$ as the other cases are completely analogous. Let $a>0$ and $\bar{\mathbf{s}}\geq\mathbf{0}$ be the initial datum (here we do not need strict positivity of $\bar{\mathbf{s}}$). We assume that $\bar{\mathbf{s}}\neq\mathbf{0}$ to avoid degeneracy (it is clear that if $\bar{\mathbf{s}}=\mathbf{0}$ there is only one admissible strategy and the problem of existence is trivial). Take a sequence $(\mathbf{x}_n,c_n)_{n\in\mathbb{N}}\subseteq\mathcal{A}\left(\bar{\mathbf{s}}\right)$ of production-consumption strategies such that $U_\sigma\left(c_n\right)\nearrow V(\bar{\mathbf{s}})$ as $n\to+\infty$. Then it is clear that, for every $n\in\mathbb{N}$ and $t\geq 0$, we have

$$\mathbf{s}_{nt}^T=\bar{\mathbf{s}}^T+\int_0^t\mathbf{x}_{ns}^T\left[\mathbf{I}-\delta\mathbf{A}\right]ds-\int_0^t c_{ns}\mathbf{e}_1^T ds\qquad\mathbf{s}_{nt}^T\geq\mathbf{x}_{nt}^T\mathbf{A}\qquad(82)$$

By estimate (71) we know that the functions $t \to f_n(t) = c_{nt}^{1-\sigma}$ belong to the space $L^{1/(1-\sigma)}_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty)$ i.e. the space of functions: $(0,+\infty) \mapsto \mathbb{R}$ that, elevated to $1/(1-\sigma)$ and multiplied by the weight function $e^{-\left(\lambda_{PF}^{-1}-\delta_{\mathbf{x}}\right)t}$ are integrable. Moreover, denoting by $||\cdot||_{1/(1-\sigma),\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}$ the norm in this space we have that $||f_n||_{1/(1-\sigma),\lambda_{PF}^{-1}-\delta_{\mathbf{x}}} \leq K\bar{\mathbf{s}}^T\mathbf{v}_{PF}$ for a suitable $K > 0$ independent of n. It follows that (by weak compactness theorems, see e.g. [8, Ch. 4]), on a subsequence (that we still denote by f_n for simplicity of notation) we have $f_n \to f_0$ weakly in $L^{1/(1-\sigma)}_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty)$, for a suitable $f_0 \in L^{1/(1-\sigma)}_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty)$. Let us call $c_0 = f_0^{1/(1-\sigma)}$. Clearly $c_0 \in L^1_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty)$. Similarly by estimate (70) we know that the functions $\mathbf{x}_n \in L^\infty_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty;\mathbb{R}^n)$ and that $||\mathbf{x}_n||_{\infty,\lambda_{PF}^{-1}-\delta_{\mathbf{x}}} \leq \lambda_{PF}^{-1}\bar{\mathbf{s}}^T\mathbf{v}_{PF}$. So, as before, there exists $\mathbf{x}_0 \in L^\infty_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty;\mathbb{R}^n)$ such that $\mathbf{x}_n \to \mathbf{x}_0$ weakly star in $L^\infty_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty;\mathbb{R}^n)$. We prove that the strategy $(\mathbf{x}_0, c_0)$ is admissible and optimal.

First it is clear that $\mathbf{x}_0 \geq \mathbf{0}$, and $c_0 \geq 0$, since the above convergencies preserve the sign constraints on the limit (see e.g. [8, Ch. 4]).

Second, consider the associated state trajectory $\mathbf{s}_0$. It is clear that

$$\mathbf{s}_{0t}^T = e^{-\delta_{\mathbf{z}}t}\bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{0s}^T\left[\mathbf{I}-\delta\mathbf{A}\right]ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)}c_{0s}\mathbf{e}_1^T ds$$

Moreover, by definition of weak star convergence in $L^\infty_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty;\mathbb{R}^n)$ we have that

$$\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{0s}^T\left[\mathbf{I}-\delta\mathbf{A}\right]ds = \lim_{n\to+\infty}\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}\mathbf{x}_{ns}^T\left[\mathbf{I}-\delta\mathbf{A}\right]ds \qquad \forall t \geq 0$$

and, by the lower semicontinuity of convex functions with respect to the the weak convergence in $L^{1/(1-\sigma)}_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty)$,

$$\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}c_{0s}ds \leq \liminf_{n\to+\infty}\int_0^t e^{-\delta_{\mathbf{z}}(t-s)}c_{ns}ds$$

so that, by (82) we have, for almost every $t \geq 0$

$$\mathbf{s}_{0t}^T \geq \limsup_{n\to+\infty}\mathbf{s}_{nt}^T \geq \limsup_{n\to+\infty}\mathbf{x}_{nt}^T\mathbf{A} \geq \mathbf{x}_{0t}^T\mathbf{A}$$

where in the last inequality we have still used the properties of the weak star convergence in $L^\infty_{\lambda_{PF}^{-1}-\delta_{\mathbf{x}}}(0,+\infty;\mathbb{R}^n)$. This gives admissibility of $(\mathbf{x}_0, c_0)$. The optimality easily follows by the concavity of U_σ which implies the weak upper semicontinuity, so that

$$\sup_{c\in\mathcal{A}(\bar{\mathbf{s}})} U_\sigma(c) = \limsup_{n\to+\infty} U_\sigma(c_n) \leq U_\sigma(c_0)$$

Finally the uniqueness property follows from the strict concavity of U_σ. ∎

The statement of Theorem 3.1 follows then from Corollary A.4, Corollary A.5 and Lemma A.7. ∎

B The value function

Here we study the problem by the dynamic programming method obtaining some results that are needed to prove the optimality conditions in the form given in Theorems 5.1 and 5.3 in Section 5 (see e.g. [3] or [36] for an introduction to the subject) and for the steady states' classification theorem in Subsection 6.4 . We will not give all the proof, for brevity. A complete mathematical treatment will be the subject of a more mathematical paper.

We start by a simple result about the class $\mathcal{A}(\bar{\mathbf{s}})$ of admissible trajectories starting at a given $\bar{\mathbf{s}}$. Recall that

- given a subset $\mathcal{A}$ of some real vector space and $\alpha \in \mathbb{R}$ we define

$$\alpha\mathcal{A} := \{\alpha a : \; a \in \mathcal{A}\} ;$$

- given subsets $\mathcal{A}, \mathcal{B}$ of some real vector space and $\alpha, \beta \in \mathbb{R}$ we define

$$\mathcal{A} + \mathcal{B} := \{a + b : \; a \in \mathcal{A}, \; b \in \mathcal{B}\}$$

$$\alpha\mathcal{A} + \beta\mathcal{B} := \{\alpha a + \beta b : \; a \in \mathcal{A}, \; b \in \mathcal{B}\}$$

Lemma B.1 *Let Assumptions 2.2, 2.3 hold. Then $\mathcal{A}(\bar{\mathbf{s}})$ is a closed and convex subset of $L^\infty_{\text{loc}}(0, +\infty; \mathbb{R}^n) \times L^1_{\text{loc}}(0, +\infty; \mathbb{R})$. Moreover, for $\alpha > 0$, $\bar{\mathbf{s}} \in \mathbb{R}^n_+$*

$$\mathcal{A}(\alpha\bar{\mathbf{s}}) = \alpha\mathcal{A}(\bar{\mathbf{s}})$$

and, for every $\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2 \in \mathbb{R}^n_+$, $\alpha \in (0, 1)$

$$\bar{\mathbf{s}}_1 \le \bar{\mathbf{s}}_2 \Longrightarrow \mathcal{A}(\bar{\mathbf{s}}_1) \subseteq \mathcal{A}(\bar{\mathbf{s}}_2) \tag{83}$$

$$\mathcal{A}(\bar{\mathbf{s}}_1) + \mathcal{A}(\bar{\mathbf{s}}_2) \subseteq \mathcal{A}(\bar{\mathbf{s}}_1 + \bar{\mathbf{s}}_2) \tag{84}$$

$$\mathcal{A}(\bar{\mathbf{s}}_1) \cup \mathcal{A}(\bar{\mathbf{s}}_2) \subseteq \mathcal{A}(\bar{\mathbf{s}}_1 + \bar{\mathbf{s}}_2)$$

$$\alpha\mathcal{A}(\bar{\mathbf{s}}_1) + (1-\alpha)\mathcal{A}(\bar{\mathbf{s}}_2) \subseteq \mathcal{A}(\alpha\bar{\mathbf{s}}_1 + (1-\alpha)\bar{\mathbf{s}}_2) \tag{85}$$

Proof. We omit the proof, since it is immediate from the definitions. ■

The value function for the problem (P_σ) has been defined in (7). The following proposition gives some of its properties.

Proposition B.2 *Let Assumptions 2.2, 2.3, 2.7 hold. Then*

(i) For every $\bar{\mathbf{s}} > \mathbf{0}$ we have for $\sigma \in (0, 1)$

$$0 \le V(\bar{\mathbf{s}}) \le \frac{\rho}{1-\sigma}\frac{\Gamma(1+\sigma)}{a^{1+\sigma}}\left[\bar{\mathbf{s}}^T\mathbf{v_{PF}}\right]^{1-\sigma}$$

while, for $\sigma = 1$

$$-\infty < V(\bar{\mathbf{s}}) \le \rho\int_t^{+\infty} e^{-\rho s} s\left[\left(\lambda_{PF}^{-1} - \delta_{\mathbf{x}}\right)s + \log\frac{\bar{\mathbf{s}}^T\mathbf{v}_{PF}}{s}\right]ds$$

and, for $\sigma \in (1, +\infty)$

$$-\infty < V(\bar{\mathbf{s}}) \le \frac{\rho}{1-\sigma}\frac{\Gamma(1+\sigma)}{a^{1+\sigma}}\left[\bar{\mathbf{s}}^T\mathbf{v_{PF}}\right]^{1-\sigma} \le 0$$

(ii) V is increasing in the sense that

$$\bar{\mathbf{s}}_1 \le \bar{\mathbf{s}}_2 \Longrightarrow V(\bar{\mathbf{s}}_1) \le V(\bar{\mathbf{s}}_2) \quad \forall \bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2 \in \mathbb{R}^n_+$$

(iii) For $\sigma \neq 1$ V is $(1-\sigma)$-homogeneous in the sense that

$$V(\alpha \bar{\mathbf{s}}) = \alpha^{1-\sigma} V(\bar{\mathbf{s}}) \quad \forall \alpha > 0,\ \bar{\mathbf{s}} \in \mathbb{R}^n_+ \tag{86}$$

and for $\sigma = 1$

$$V(\alpha \bar{\mathbf{s}}) = \rho^{-1} \log \alpha + V(\bar{\mathbf{s}}) \quad \forall \alpha > 0,\ \bar{\mathbf{s}} \in \mathbb{R}^n_+ \tag{87}$$

(iv) V is locally Lipschitz continuous on the open positive orthant $Int\mathbb{R}^n_+$ (and continuous at the boundary if $\sigma \in (0,1)$) and concave.

(v) V is two times differentiable a.e. on $\mathbb{R}^n_+$ and $\nabla V \ge 0$ at every point of differentiability (the first component is strictly positive). Moreover V admits non-empty superdifferential at every point of $Int\mathbb{R}^n_+$ and $D^+V(\bar{\mathbf{s}}) \subseteq \mathbb{R}^n_+$ for every $\bar{\mathbf{s}} \in Int\mathbb{R}^n_+$.

Proof.

Proof of (i). This is a consequence of the Assumption 2.7 and easily follows by the Lemma A.2.

Proof of (ii). This a direct consequence of (83).

Proof of (iii). Since $\mathcal{A}(\alpha \bar{\mathbf{s}}) = \alpha \mathcal{A}(\bar{\mathbf{s}})$ we have, when $\sigma \neq 1$,

$$V(\alpha \bar{\mathbf{s}}) = \sup_{(\mathbf{x},c) \in \mathcal{A}(\alpha \bar{\mathbf{s}})} \int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt = \sup_{(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}})} \int_0^{+\infty} e^{-\rho t} \frac{[\alpha c_t]^{1-\sigma}}{1-\sigma} dt = \alpha^{1-\sigma} V(\bar{\mathbf{s}})$$

The case when $\sigma = 1$ is completely analogous and therefore omitted.

Proof of (iv). For $\alpha \in (0,1)$ we have

$$\begin{aligned}
& \alpha V(\bar{\mathbf{s}}_1) + (1-\alpha) V(\bar{\mathbf{s}}_2) \\
= \ & \alpha \sup_{(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}}_1)} \int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt + (1-\alpha) \sup_{(\mathbf{x},c) \in \mathcal{A}(\bar{\mathbf{s}}_2)} \int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt \\
= \ & \sup_{\substack{(\mathbf{x}_1,c_1) \in \mathcal{A}(\bar{\mathbf{s}}_1), \\ (\mathbf{x}_2,c_2) \in \mathcal{A}(\bar{\mathbf{s}}_2)}} \int_0^{+\infty} e^{-\rho t} \frac{\alpha c_{1t}^{1-\sigma} + (1-\alpha) c_{2t}^{1-\sigma}}{1-\sigma} dt \\
\le \ & \sup_{\substack{(\mathbf{x}_1,c_1) \in \mathcal{A}(\bar{\mathbf{s}}_1), \\ (\mathbf{x}_2,c_2) \in \mathcal{A}(\bar{\mathbf{s}}_2)}} \int_0^{+\infty} e^{-\rho t} \frac{[\alpha c_{1t} + (1-\alpha) c_{2t}]^{1-\sigma}}{1-\sigma} dt
\end{aligned}$$

where in the last inequality we have used the concavity of the instantaneous utility u_σ. The latter implies, because of (85), that

$$\alpha V(\bar{\mathbf{s}}_1) + (1-\alpha) V(\bar{\mathbf{s}}_2) \le \sup_{(\mathbf{x},c) \in \mathcal{A}(\alpha \bar{\mathbf{s}}_1 + (1-\alpha)\bar{\mathbf{s}}_2)} \int_0^{+\infty} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$

$$= V\left(\alpha \bar{\mathbf{s}}_1 + (1-\alpha)\bar{\mathbf{s}}_2\right)$$

which gives the concavity.

The Lipschitz continuity in the interior follows by applying standard results on concave functions (see e.g. [12], [29]) while on the boundary one needs to use the definition of V and argue as in the proof of Lemma A.7; this part is not trivial but we omit it for brevity.

Proof of (v). This part follows directly from the concavity via the so-called Alexandroff Theorem (see e.g. [13]) and from the monotonicity via the definition of subdifferential. ■

The following Proposition is a standard version of a general result for control problems and is known as the Bellman Optimality Principle or Dynamic Programming Principle (see e.g. [6], or [3, 15],).

Proposition B.3 *Let Assumptions 2.2 ,2.3, 2.7 hold. For every $t \geq 0$ we set $\mathcal{A}_t(\bar{\mathbf{s}})$ as the set of control strategies that satisfies all the constraints on $(\mathbf{x}, c)$ and $\mathbf{s}$ up to time t, and*

$$J_t(\mathbf{x}, c) = \int_0^t e^{-\rho s} \frac{c_s^{1-\sigma}}{1-\sigma} ds + e^{-\rho t} V\left(\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x},c)}\right)$$

Then, for every $(\mathbf{x}, c) \in \mathcal{A}(\bar{\mathbf{s}})$ the function $t \to g(t) = J_t(\mathbf{x}, c)$ is nonincreasing and we have, for every $t \geq 0$

$$V(\bar{\mathbf{s}}) = \sup_{(\mathbf{x},c)\in\mathcal{A}_t(\bar{\mathbf{s}})} J_t(\mathbf{x}, c) \tag{88}$$

Moreover, if $(\mathbf{x}, c)$ is optimal for (P_σ) then its restriction to $[0, t]$ is optimal for the problem $(P_{t,\sigma})$ of maximizing $J_t(\mathbf{x}, c)$ and the function $t \to g(t)$ is constant.

Proof. The proof is standard (see e.g. [3]) and we omit it. ■

The Hamilton-Jacobi equation associated with our problem is

$$\rho u(\mathbf{s}) = H_0(\mathbf{s}, \nabla u(\mathbf{s})) \qquad \forall \mathbf{s} \geq 0. \tag{89}$$

where we recall that the maximum value Hamiltonian H_0 is given in Section 4.

Using the Dynamic Programming Principle (88) and some regularity assumption on the problem one can prove (sse e.g. [13, 30]) that the value function V is the unique solution of the above equation in the sense of viscosity solutions. However these assumptions are not verified in this context, so a weaker result holds (see [16] and also [4] for related results). Anyway in this work we are not interested in this problem. We state below what we need (see [16, 10] for a proof).

Proposition B.4 *Let Assumptions 2.2 ,2.3, 2.7 hold. Then the value function V is a viscosity solution of the equation (89) in the sense introduced in [18]. Moreover it is also a bilateral solution in the sense defined in [3, p. 133]. In particular, for every $\mathbf{s} > \mathbf{0}$, $\mathbf{a} \in D^+V(\mathbf{s})$ we have*

$$\rho V(\mathbf{s}) = H_0(\mathbf{s}, \mathbf{a})$$

Moreover V is always continuously differentiable with respect to the first variable when $\mathbf{s}^T\mathbf{e}_1 > 0$. In the case when $n = 2$, V is continuously differentiable on $Int\mathbb{R}^2_+$.

We observe that (see e.g. [3, p. 133]) from the dynamic programming principle (Proposition B.3) the following optimality condition follows: a control strategy $(\hat{\mathbf{x}},\hat{c}) \in \mathcal{A}(\overline{\mathbf{s}})$ is optimal for the problem (P_σ) if and only if the function

$$g(t) = \int_0^t e^{-\rho s} \frac{\hat{c}_s^{1-\sigma}}{1-\sigma} ds + e^{-\rho t} V(\hat{\mathbf{s}}_t)$$

is nondecreasing for $t \geq 0$. This fact, together with Proposition B.4 above implies the following necessary condition of optimality.

Proposition B.5 *Let Assumptions 2.2 ,2.3, 2.5 and 2.7 hold. Assume also that* $(\hat{\mathbf{x}},\hat{c}) \in \mathcal{A}(\overline{\mathbf{s}})$ *is optimal for the problem* (P_σ) *and let* $\hat{\mathbf{s}}$ *be the corresponding optimal state. Then, for a.e.* $t \geq 0$, *for every* $\mathbf{a} \in D^+V(\hat{\mathbf{s}}_t)$

$$\left[\frac{\hat{c}_s^{1-\sigma}}{1-\sigma} - \rho V(\hat{\mathbf{s}}_t) + \left(\dot{\hat{\mathbf{s}}}_t\right)^T \mathbf{a}\right] = 0$$

i.e.

$$\hat{\mathbf{x}}_t^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{a} - \hat{c}_t \mathbf{e}_1^T \mathbf{a} + \frac{\hat{c}_t^{1-\sigma}}{1-\sigma} = \rho V(\hat{\mathbf{s}}_t)$$

and also

$$\hat{\mathbf{x}}_t^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{a} = \sup_{\mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \mathbf{A} \leq \mathbf{s}_t^T} \left\{ \mathbf{x}^T [\mathbf{I} - \delta \mathbf{A}] \mathbf{a} \right\} \tag{90}$$

$$-\hat{c}_t \mathbf{e}_1^T \mathbf{a} + \frac{\hat{c}_t^{1-\sigma}}{1-\sigma} = \sup_{c \geq 0} \left\{ -c \mathbf{e}_1^T \mathbf{a} + \frac{c^{1-\sigma}}{1-\sigma} \right\}. \tag{91}$$

Proof. It follows using Proposition B.2 and applying the same argument of [4] and [33] (see also [3, p.133-136]) adapted to this case. We omit it for brevity. ■

The following corollary will be useful in proving optimality conditions.

Corollary B.6 *Let Assumptions 2.2 ,2.3, 2.5 and 2.7 hold. Assume also that* $(\hat{\mathbf{x}},\hat{c}) \in \mathcal{A}(\overline{\mathbf{s}})$ *is optimal for the problem* (P_σ) *and let* $\hat{\mathbf{s}}$ *be the corresponding optimal state. Then, we have*

(i) for every $t \geq 0$ $\hat{c}_t$ *is continuous and strictly positive and*

$$\hat{c}_t^{-\sigma} = D_1 V(\hat{\mathbf{s}}_t)$$

(ii) for every $t \geq 0$

$$\hat{\mathbf{s}}_t > 0.$$

Proof. The point (i) follows from the above Proposition B.5 and from the continuous differentiability of V in the direction $\mathbf{e_1}$.

To prove the point (ii) we look first at the goods not used for consumption: the fact that $\hat{\mathbf{s}}_t^T \mathbf{e}_j > 0$ for $j = 2, ..., n$ is an obvious consequence of the state equation and of the constraints and holds for every admissible trajectory. Indeed

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T [\mathbf{I} - \delta \mathbf{A}] - \delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T; \qquad \mathbf{s}_0 = \overline{\mathbf{s}}$$

with the constraints

$$\mathbf{x}_t \geq \mathbf{0}; \qquad \mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}; \qquad c_t \geq 0.$$

implies that, if $\delta \geq 0$

$$\dot{\mathbf{s}}_t^T \geq \mathbf{x}_t^T \mathbf{I} - \delta_{\mathbf{x}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T; \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

which gives, for $j = 2, ..., n$,

$$\dot{\mathbf{s}}_t^T \mathbf{e}_j \geq -\delta_{\mathbf{x}} \mathbf{s}_t^T \mathbf{e}_j$$

so that

$$\mathbf{s}_t^T \mathbf{e}_j \geq e^{-\delta_{\mathbf{x}} t} \bar{\mathbf{s}}^T \mathbf{e}_j.$$

Now, if $\delta < 0$ then we have $-\delta \mathbf{x}^T \mathbf{A} \geq \mathbf{0}$ so that

$$\dot{\mathbf{s}}_t^T \geq -\delta_{\mathbf{z}} \mathbf{s}_t^T - c_t \mathbf{e}_1^T; \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

which gives, for $j = 2, ..., n$,

$$\mathbf{s}_t^T \mathbf{e}_j \geq e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}}^T \mathbf{e}_j.$$

and the claim follows.

Regarding the first component $\hat{\mathbf{s}}_t^T \mathbf{e}_1$ we first observe that the constraint $\mathbf{s} \geq \mathbf{x}^{\mathbf{T}} \mathbf{A}$ and the Assumption 2.6 gives, for every admissible trajectory $\mathbf{x}_t^T \mathbf{e}_1 \leq \left(\mathbf{s}_t^T \mathbf{e}_1 \right) / a_{11}$. Since the state equation gives

$$\dot{\mathbf{s}}_t^T \leq \mathbf{x}_t^T \left[\mathbf{I} - \delta_{\mathbf{x}} \mathbf{A} \right] - c_t \mathbf{e}_1^T; \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

then

$$\dot{\mathbf{s}}_t^T \mathbf{e}_1 \leq \left(\mathbf{s}_t^T \mathbf{e}_1 \right) / a_{11} - \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A} \mathbf{e}_1 - c_t$$

so that, for every $t_1 < t_2$

$$\mathbf{s}_{t_2}^T \mathbf{e}_1 \leq e^{(t_2 - t_1)/a_{11}} \mathbf{s}_{t_1}^T \mathbf{e}_1.$$

This implies that, if $\mathbf{s}_{t_1}^T \mathbf{e}_1 = 0$, then $\mathbf{s}_{t_2}^T \mathbf{e}_1 = 0$ for every $t_2 > t_1$. This implies also that $c = 0$ after t_1. This behavior is not admissible for $\sigma \geq 1$. Moreover for $\sigma \in (0, 1)$ it cannot be optimal due to part (i) of this corollary. ■

C Proof of Theorems 5.1 and 5.3

Here we prove Theorems 5.1 and 5.3 dividing the proof in two subsections: the first about sufficiency and the second about necessity.

C.1 Sufficient Conditions

To prove sufficient conditions we will suppose that Assumptions 2.2, 2.3 and 2.7 hold troughout all this subsection even if, as noted in Remark 5.2, they can be considerably relaxed without big effort. We will not assume 2.5 and 2.6 since they are not needed here. In fact our sufficient conditions holds under weaker Assumptions than the necessary ones,also because they do not need the co-state inclusion (25) to establish sufficiency but only a consequence of it: the nonnegativity of $\mathbf{v}$ (that follows from the monotonicity of V).

The proof is given below and follows the method of [32, p. 385, Theorem 11]. However, the assumptions here are different, due to the singularity at 0 of the istantaneous utility and to the assumptions on the control strategy. The modifications are quite straightforward: we give the proof for the reader's convenience.

Proof of Theorem 5.1. Let $(\hat{\mathbf{x}}, \hat{c})$ be the above admissible production-consumption strategy, let $\hat{\mathbf{s}}$ be the associated commodities' stock trajectory. Let $(\mathbf{x}, c)$ be another admissible strategy starting at the same point $\overline{\mathbf{s}}$. Consider, the quantity

$$\Delta = U_\sigma(\hat{c}) - U_\sigma(c) = \int_0^{+\infty} e^{-\rho t}\left[\frac{\hat{c}_t^{1-\sigma} - c_t^{1-\sigma}}{1-\sigma}\right] dt.$$

We want to prove that $\Delta \geq 0$, for every $(\mathbf{x}, c)$. To avoid a problem with integrability at infinity we will take for every $\theta > 0$ the quantity

$$\Delta_\theta = \int_0^{\theta} e^{-\rho t}\left[\frac{\hat{c}_t^{1-\sigma} - c_t^{1-\sigma}}{1-\sigma}\right] dt.$$

Then, by the definition of Hamiltonian we have

$$\begin{aligned} \Delta_\theta &= \int_0^\theta e^{-\rho t}\left[\left(H\left(\hat{\mathbf{s}}_t, \mathbf{v}_t; \hat{\mathbf{x}}_t, \hat{c}_t\right) - H\left(\mathbf{s}_t, \mathbf{v}_t; \mathbf{x}_t, c_t\right)\right)\right. \\ &\quad \left. -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_t + \mathbf{x}_t^T\left[\mathbf{I} - \delta\mathbf{A}\right]\mathbf{v}_t - c_t \mathbf{e}_1^T \mathbf{v}_t - \left(-\delta_{\mathbf{z}}^T \hat{\mathbf{s}}_t^T \mathbf{v}_t + \hat{\mathbf{x}}_t^T\left[\mathbf{I} - \delta\mathbf{A}\right]\mathbf{v}_t - \hat{c}_t \mathbf{e}_1^T \mathbf{v}_t\right)\right] dt \\ &= \int_0^\theta e^{-\rho t}\left[H\left(\hat{\mathbf{s}}_t, \mathbf{v}_t; \hat{\mathbf{x}}_t, \hat{c}_t\right) - H\left(\mathbf{s}_t, \mathbf{v}_t; \mathbf{x}_t, c_t\right) + \left(\dot{\mathbf{s}}_t - \dot{\hat{\mathbf{s}}}_t\right)^T \mathbf{v}_t\right] dt. \end{aligned} \tag{92}$$

Now we recall that the conditions (12)—(15) imposed in the statement of the theorem imply that, for almost every $t \geq 0$

$$\begin{aligned} \hat{\mathbf{x}}_t &\in \operatorname{argmax}\left\{\mathbf{x}^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v}_t;\ \mathbf{x} \geq 0;\ \mathbf{x}^T\mathbf{A} \leq \hat{\mathbf{s}}_t^T\right\}; \\ \hat{c}_t &\in \operatorname{argmax}\left\{-c\mathbf{e}_1^T\mathbf{v} + \frac{c^{1-\sigma}}{1-\sigma};\quad c \geq 0\right\} \end{aligned} \tag{93}$$

which gives that, for almost every $t \geq 0$,

$$\begin{aligned} H\left(\hat{\mathbf{s}}_t, \mathbf{v}_t; \hat{\mathbf{x}}_t, \hat{c}_t\right) &= H_0(\hat{\mathbf{s}}_t, \mathbf{v}_t) = H_{01}(\hat{\mathbf{s}}_t, \mathbf{v}_t) + H_{02}(\hat{\mathbf{s}}_t, \mathbf{v}_t) + H_{03}(\mathbf{v}_t) \\ &= -\delta_{\mathbf{z}}^T \hat{\mathbf{s}}_t^T \mathbf{v}_t + \hat{\mathbf{x}}_t^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v}_t + \frac{\sigma}{1-\sigma}\left(\mathbf{e}_1^T\mathbf{v}_t\right)^{\frac{\sigma-1}{\sigma}} \end{aligned}$$

so that

$$\begin{aligned} & H\left(\hat{\mathbf{s}}_t, \mathbf{v}_t; \hat{\mathbf{x}}_t, \hat{c}_t\right) - H\left(\mathbf{s}_t, \mathbf{v}_t; \mathbf{x}_t, c_t\right) \\ =\ & \delta_{\mathbf{z}}\left(\mathbf{s}_t - \hat{\mathbf{s}}_t\right)^T \mathbf{v}_t + \hat{\mathbf{x}}_t^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v}_t - \mathbf{x}_t^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v}_t + H_{03}(\mathbf{v}_t) - H_3(\mathbf{v}_t, c_t). \end{aligned}$$

Now we observe that, by (93), $H_{03}(\mathbf{v}_t) - H_3(\mathbf{v}_t, c_t) \geq 0$ for almost every $t \geq 0$. Moreover by the (13), (14) and (15) condition, for almost every $t \geq 0$

$$\hat{\mathbf{x}}_t^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v}_t = \hat{\mathbf{x}}_t^T\mathbf{A}\mathbf{q}_t = \hat{\mathbf{s}}_t^T\mathbf{q}_t$$

$$\mathbf{x}_t^T[\mathbf{I}-\delta\mathbf{A}]\mathbf{v}_t \leq \mathbf{x}_t^T\mathbf{A}\mathbf{q}_t \leq \mathbf{s}_t^T\mathbf{q}_t$$

which gives for a.e. $t \geq 0$.

$$H\left(\hat{\mathbf{s}}_t, \mathbf{v}_t; \hat{\mathbf{x}}_t, \hat{c}_t\right) - H\left(\mathbf{s}_t, \mathbf{v}_t; \mathbf{x}_t, c_t\right) \geq -\delta_{\mathbf{z}}\left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right)^T \mathbf{v}_t + \left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right)^T \mathbf{q}_t$$

Putting this last inequality into (92) we get

$$\Delta_\theta \geq \int_0^\theta e^{-\rho t}\left[\left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right)^T \mathbf{q}_t - \delta_{\mathbf{z}}\left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right)^T \mathbf{v}_t + \left(\dot{\mathbf{s}}_t - \dot{\hat{\mathbf{s}}}_t\right)^T \mathbf{v}_t\right] dt. \tag{94}$$

Integrating by parts we get (using that $\frac{d}{dt}\mathbf{v}_t e^{-\rho t} = e^{-\rho t}(-\rho \mathbf{v}_t + \dot{\mathbf{v}}_t)$)

$$\begin{aligned}\int_0^\theta e^{-\rho t}\left(\dot{\mathbf{s}}_t - \dot{\hat{\mathbf{s}}}_t\right)^T \mathbf{v}_t dt \; &= \; \left[\left(\mathbf{s}_t - \hat{\mathbf{s}}_t\right)^T \mathbf{v}_t e^{-\rho t}\right]_0^\theta - \int_0^\theta \left(\mathbf{s}_t - \hat{\mathbf{s}}_t\right)^T \left(\frac{d}{dt}\left(\mathbf{v}_t e^{-\rho t}\right)\right) dt \\ &= \left(\mathbf{s}_\theta - \hat{\mathbf{s}}_\theta\right)^T \mathbf{v}_\theta e^{-\rho\theta} + \int_0^\theta e^{-\rho t}\left[\left(\mathbf{s}_t - \hat{\mathbf{s}}_t\right)^T \mathbf{q}_t - \delta_{\mathbf{z}}\left(-\hat{\mathbf{s}}_t + \mathbf{s}_t\right)^T \mathbf{v}_t\right] dt\end{aligned}$$

which implies, by (94)

$$\Delta_\theta \geq \left(\mathbf{s}_\theta - \hat{\mathbf{s}}_\theta\right)^T \mathbf{v}_\theta e^{-\rho\theta}$$

which gives, thanks to the positivity of every admissible state trajectory $\mathbf{s}_\theta$, to condition (16) (the so-called transversality condition) and to (17) (the nonnegativity of the prices $\mathbf{v}$)

$$\Delta = \lim_{\theta \to +\infty} \Delta_\theta \geq \limsup_{\theta \to +\infty} \left(\mathbf{s}_\theta - \hat{\mathbf{s}}_\theta\right)^T \mathbf{v}_\theta e^{-\rho\theta} \geq 0$$

and the claim follows. ■

As we already said, the above theorem is a modified and simplified version of general sufficient conditions (see e.g. [32, p. 385-389]). It holds true also if we assume that the co-state $\mathbf{v}$ is only piecewise continuous and piecewise differentiable putting suitable conditions on the jump points.

C.2 Necessary Conditions

We pass now to necessary conditions giving a version of the so-called Maximum Principle (introduced first in [26]) adapted to our case. Since we are working with a non-standard problem in optimal control theory, the Theorem 5.3 does not follow directly from results known in the literature, even if its statement is similar to some of them (we recall in particular [25, Theorem VI.3.108], [32, Theorem 9 p. 381], [31] and [1]). However a complete proof of Theorem 5.3 would be very long and technical. For this reason here we will only sketch the proof focusing on the main differences with [25, Theorem VI.3.108] and [32, Theorem 9 p. 381] (whose proof is given in [31]).

We point out that [32, Theorem 9 p. 381] assumes that the classical "constraints qualifications" hold to get absolute continuity of $\mathbf{v}$ and to avoid $\mathbf{q}$ being a measure. These classical "constraints qualifications" do not hold in our case, but, using Assumptions 2.5

and 2.6 we get the positivity of the trajectories that still allows to rule out the measure case (see [25, Lemma VI.3.100]).

Proof of Theorem 5.3.

First part: proof of (18)–(23).

To prove (18)–(23) we follow the line of the proof of [32, Theorem 9 p. 381] (whose proof is given in [31] and uses previous results of [25, Section VI.3] for the finite horizon case) except for the following facts:

1. In [32, Theorem 9 p. 381] it is assumed that the optimal control is locally bounded. In our case we know that $\mathbf{x}$ is locally bounded (thanks to the constraint $\mathbf{x}^T\mathbf{A} \leq \mathbf{s}$) but c is only locally integrable. So we need to prove that c is locally bounded.

2. In [32, Theorem 9 p. 381] it is assumed that the so-called "constraints qualifications" hold. They do not hold in our case, so we need to substitute them with others ([25, Hypothesis VI.3.98]).

3. The Hamiltonian can be degenerate, i.e. it is defined as

$$H\left(\mathbf{s}, v^0, \mathbf{v}; \mathbf{x}, c\right) = -\delta_{\mathbf{z}}\mathbf{s}^T\mathbf{v} + \mathbf{x}^T[\mathbf{I} - \delta\mathbf{A}]\mathbf{v} - c\mathbf{e}_1^T\mathbf{v} + v^0\frac{c^{1-\sigma}}{1-\sigma} \qquad \mathbf{s}, \mathbf{v}, \mathbf{x}, \in \mathbb{R}^n;$$

   where v^0 can be 0 or 1. We need to prove that in our case we can set $v^0 = 1$.

We now briefly show how to prove the above facts in our case.

Proof of 1). It follows from Corollary B.6, part (i). To deal with this lack of local boundedness one could also use the approach exposed in [11, p. 167, condition c'] (see also the footnotes at p. 276, 372 in [32]); we have chosen the dynamic programming approach since it gives rise to other useful results for the study of our problem. For the proof of optimality conditions via dynamic programming in the case of optimal control problems without state/control constraints one could see e.g. [3]; however we are not aware of such kind of results for the case with state/control constraints. (see Appendix B).

Proof of 2). First we observe that the "constraints qualifications" do not hold here. In fact they require, in our case, that the function:

$$[0, +\infty) \mapsto \mathbb{R}^{2n+1}; \qquad t \mapsto (\hat{\mathbf{s}}_t, \hat{\mathbf{x}}_t, \hat{c}_t)$$

satisfies the $2n+1$ constraints

$$\hat{\mathbf{x}}_t \geq \mathbf{0}; \qquad \hat{\mathbf{s}}_t^T - \hat{\mathbf{x}}_t^T\mathbf{A} \geq \mathbf{0}; \qquad \hat{c}_t \geq 0$$

with at most n equalities for a.e. $t \geq 0$. Even if $\hat{\mathbf{s}}_t > \mathbf{0}$, $\hat{c}_t > 0$ for a.e. $t \geq 0$ this is not guaranteed in general. It is enough to have some overdetermined constraints in the linear programming problem (8).

Instead, using that, from Corollary B.6, $\hat{\mathbf{s}}_t > \mathbf{0}$, $\hat{c}_t > 0$ for a.e. $t \geq 0$, we get that [25, Hypothesis VI.3.98] holds. Indeed this latter asks that there exists an admissible production strategy $\mathbf{x}$ such that, for any $t \geq 0$, $\mathbf{x}_t^T\mathbf{A} < \hat{\mathbf{s}}_t$ which is clearly true in our case. Then [25, Lemma VI.3.100] hold true and so $\mathbf{v}$ is absolutely continuous.

Proof of 3). This is guaranteed by the strict positivity of the optimal control c coming from Proposition 5.7. In fact, if $v^0 = 0$ then the Hamiltonian H_3 would be

$$H_3(\mathbf{v}, c) = -c\,\mathbf{e}_1^T \mathbf{v}$$

whose maximum point is always $c = 0$. Since $\hat{c}_t \in \operatorname{argmax}\{H_3(\mathbf{v}; c);\ c \geq 0\}$ this would contradict the optimality of $\hat{c}$.

Second part: proof of (24) and (25).

The necessity of the transversality condition (24) and of the co-state inclusion (25) is not proved in [32, Theorem 9 p.381] nor is it contained in other results in the literature.

We first show how to prove the co-state inclusion: by the Dynamic Programming Principle we know that for every $t \geq 0$ the restriction of $(\hat{\mathbf{x}}, \hat{c})$ to $[0, t]$ is optimal for the problem $(P_{t,\sigma})$ of maximizing the functional

$$J_t(\bar{\mathbf{s}}; \mathbf{x}, c) = \int_0^t e^{-\rho s} \frac{c_s^{1-\sigma}}{1-\sigma} ds + e^{-\rho t} V\left(\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x},c)}\right)$$

under the constraints (for $s \in [0, t]$)

$$\dot{\mathbf{s}}_s^T = -\delta_{\mathbf{z}} \mathbf{s}_s^T + \mathbf{x}_s^T [\mathbf{I} - \delta \mathbf{A}] - c_s \mathbf{e}_1^T$$

$$\mathbf{x}_s \geq \mathbf{0}; \qquad \mathbf{x}_s^T \mathbf{A} \leq \mathbf{s}_s^T; \qquad c_s \geq 0$$

($(\mathbf{x}, c)$ are bounded on $[0, t]$). Then we can apply to problem $(P_{t,\sigma})$ the finite horizon results [25, Section VI.3, Theorem 93, Remark 125], and [25, Section VI.2, Theorem 27, Remark 39]. The result is that there exist two bounded functions $\mathbf{v}^t$ and $\mathbf{q}^t : [0, t] \mapsto \mathbb{R}^n$ such that $\mathbf{v}^t$ is absolutely continuous and

$$\dot{\mathbf{v}}_s^t = (\rho + \delta_{\mathbf{z}}) \mathbf{v}_s^t - \mathbf{q}_s^t; \text{ on } [0, t];$$

$$\mathbf{v}_t^t \in D^+ V(\hat{\mathbf{s}}_t)$$

Then we prove that $\mathbf{v}_s^t \in D^+ V(\hat{\mathbf{s}}_s)$ for a.e. $s \in [0, t]$. This is quite hard and we skip it, one can see e.g. [9, Theorem 3.1] for similar results but in a different context. Next we pass to the limit for $t \to +\infty$ as in [31] getting the existence of the required co-state $\mathbf{v}$. The stability property of the superdifferential implies then that (25) holds.

The transversality then follows by using an argument similar to the one of [7, Theorem 3.1] and we omit it. ■

D Two examples

In this Appendix, we prove, by means of two examples, that the set of supported steady states given in Definition 6.4 is a proper subset of the set of optimal steady states. Essentially, the examples show that the necessary conditions in Theorem 5.3 cannot be extended to the boundary of the first orthant even if the optimal path is a steady state in which production is semipositive. In both examples, the stock of a pure capital good is zero at t = 0 and, because of the structure of the technology, it cannot be produced at any time. Since this commodity is also needed in the production of another commodity, which exists at time 0, in both cases the upper bound of the rate of growth becomes $-\delta_x$.

D.1 All commodities required to produce the consumption good are available, but one of them cannot be produced

Consider a three-sector system with the following data:

$$\sigma = 1, \qquad \mathbf{A} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \delta_z = \delta_x < \lambda_{PF}^{-1} = \tfrac{1}{3}.$$

The Classification Theorem tells us that there are the following price supported steady states

1) For $0 < \rho < \frac{1}{3}$ we have $g = \frac{2}{3} - \delta_x - \rho$

$$\mathbf{x}_t^T = c_0 \mathbf{e}_1^T [I - (\tfrac{1}{3} - \rho)\mathbf{A}]^{-1} e^{gt}, \qquad \mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A}$$

with the supporting prices

$$\mathbf{v}_t = c_0^{-1} \mathbf{e} e^{-gt}, \qquad \mathbf{q}_t = \tfrac{1}{3} c_0^{-1} \mathbf{e} e^{-gt}$$

2) For $\frac{1}{3} \le \rho \le \frac{2}{3}$ we have

$$g = -\delta_x, \qquad \mathbf{x}_t^T = c_0 \mathbf{e}_1^T e^{gt} + \alpha \mathbf{e}_3^T e^{gt}$$

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + (\beta + t\alpha) \mathbf{e}_3^T e^{gt}$$

with the supporting prices

$$\mathbf{v}_t = \frac{c_0^{-1}}{3\rho} \begin{bmatrix} 3\rho \\ 2 - 3\rho \\ y \end{bmatrix} e^{-gt}, \qquad \mathbf{q}_t = \frac{c_0^{-1}}{3} \begin{bmatrix} 3\rho \\ 2 - 3\rho \\ y \end{bmatrix} e^{-gt}$$

where $\max\left(0, \frac{2-5\rho}{\rho}\right) \le y \le \frac{2\rho}{1-\rho}$, $\alpha = 0, \beta = 0$ for $\frac{1}{3} \le \rho < \frac{2}{5}$, $\alpha \ge 0, \beta \ge 0, \alpha y = \beta y = 0$ for $\frac{2}{5} \le \rho \le \frac{2}{3}$.

3) For $\rho > \frac{2}{3}$ we have

$$g = \tfrac{2}{3} - \delta_x - \rho, \qquad \mathbf{x}_t^T = \frac{2c_0}{3\rho} \mathbf{e}_1^T e^{gt}$$

$$\mathbf{s}_t = \frac{c_0}{\delta} \mathbf{e}_1 e^{gt} + \left[\left(\frac{c_0}{\delta} + \alpha \right) \mathbf{e}_2 + \beta \mathbf{e}_3 \right] e^{-\delta_z t}$$

with the supporting prices

$$\mathbf{v}_t = c_0^{-1} \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} e^{-gt}, \qquad \mathbf{q}_t = \frac{2c_0^{-1}}{3} \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} e^{-gt}$$

where $0 \le y \le 2$ and $\alpha y = 0$.

If we superimpose $\bar{\mathbf{s}}^T\mathbf{e}_3 = 0$, then we still have the price supported steady states defined in 2) and 3) (obviously with $\alpha = \beta = 0$ in 2) and $\beta = 0$ in 3)) but we cannot have the price supported steady state defined in 1). However if $\bar{\mathbf{s}}^T = (h, h, 0)$ and $0 \le \rho < \frac{1}{3}$, an optimal solution can be determined and this optimal solution is a steady state optimal solution. Yet, it is easily shown the there are not price- and rental-vectors supporting it. In fact, by applying the Theorem 5.1 (taking account of Remark 5.2) to the truncated problem

$$\max \int_0^{+\infty} e^{-\rho t} \log c_t dt$$

subject to

$$\dot{\tilde{\mathbf{s}}}_t^T = \tilde{x}_t \mathbf{e}_1 - \delta_z \tilde{\mathbf{s}}_t^T - c_t \mathbf{e}_1^T, \qquad \tilde{s}_0^T = (h, h)$$

$$\tilde{x}_t \ge 0, \qquad \tilde{\mathbf{s}}_t^T \ge \tilde{x}_t \left(\frac{3}{2}, \frac{3}{2}\right), \qquad c_t \ge 0$$

it is easily checked that

$$c_t = \tfrac{2}{3} h e^{gt}, \qquad \mathbf{x}_t^T = \tfrac{2}{3} h \mathbf{e}_1^T e^{gt}, \qquad \left(\tilde{\mathbf{x}}_t^T = \tfrac{2}{3} h e^{gt}\right)$$

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A}, \qquad \left(\tilde{\mathbf{s}}_t^T = \tilde{x}_t \left(\frac{3}{2}, \frac{3}{2}\right)\right)$$

with $g = -\delta_x$ is an optimal steady state solution of the original (truncated) problem and the price- and rental-vectors supporting the truncated problem are:

$$\tilde{\mathbf{v}}_t = \frac{1}{2h\rho} \begin{bmatrix} 3\rho \\ 2 - 3\rho \end{bmatrix} e^{-gt}, \qquad \tilde{\mathbf{q}}_t = \frac{1}{2h} \begin{bmatrix} 3\rho \\ 2 - 3\rho \end{bmatrix} e^{-gt}$$

It is easily checked that prices and rentals which are candidates to support the optimal solution of the original problem need to satisfy

$$\mathbf{v}_t = \frac{1}{2h\rho} \begin{bmatrix} 3\rho \\ 2 - 3\rho \\ 0 \end{bmatrix} e^{-gt} + \begin{bmatrix} 0 \\ 0 \\ y_t \end{bmatrix}, \qquad \mathbf{q}_t = \frac{1}{2h} \begin{bmatrix} 3\rho \\ 2 - 3\rho \\ 0 \end{bmatrix} e^{-gt} + \begin{bmatrix} 0 \\ 0 \\ w_t \end{bmatrix}$$

where y_t and w_t need to satisfy the inequalities

$$w_t \ge \frac{2 - 5\rho}{2h\rho} e^{-gt}, \qquad y_t \le \frac{1}{h} e^{-gt} + w_t$$

$$\dot{y}_t = (\rho + \delta_z) y_t - w_t, \qquad w_t \ge 0, \quad y_t \ge 0,$$

the last inequality being a consequence of the fact that $\mathbf{v}_t$ need to be in the superdifferential of V. Since the last but one inequality is satisfied, for $0 \le \rho < \frac{1}{3}$, once the first inequality is satisfied, it will be ignored here. The above inequalities in y_t and w_t have a solution if and only if there is a solution to the system in θ_t

$$\dot{\theta}_t \le \rho \theta_t - \frac{2 - 5\rho}{2h\rho}, \tag{95}$$

$$\dot{\theta}_t \leq (\rho - 1)\theta_t + \frac{1}{h}, \tag{96}$$

$$\theta_t \geq 0. \tag{97}$$

In fact it is enough to substitute $y_t = \theta_t e^{-gt}$ and $w_t = (\rho\theta_t - \dot{\theta})e^{-gt}$. Inequality (95) is binding for $0 \leq \theta_t \leq \frac{2-3\rho}{2h\rho}$, whereas inequality (96) is binding for $\theta_t \geq \frac{2-3\rho}{2h\rho}$. Hence inequalities (95) and (96) are both satisfied only if

$$\dot{\theta}_t \leq -\frac{2 - 7\rho + 3\rho^2}{2h\rho}. \tag{98}$$

Since, for $0 \leq \rho < \frac{1}{3}$, the right hand side of inequality (98) is negative, a function θ_t satisfying inequalities (95) and (96) for $0 \leq \rho < \frac{1}{3}$ is estimated from above by a decreasing straight line. Hence it cannot satisfy inequality (97) for each t.

D.2 The consumption good is available, but cannot be produced

Consider a three-sector system with the following data:

$$\sigma = 1, \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \delta_z > \delta_x.$$

If we superimpose $\bar{\mathbf{s}}^T \mathbf{e}_3 = 0$, then $\mathbf{x}_t^T \mathbf{e}_1 = 0$, so that no steady state mentioned in the Classification Theorem exists. However, if $\bar{\mathbf{s}}^T = (h, k, 0)$, then an optimal solution can be determined for $0 < \rho$ and this optimal solution is a steady state optimal solution, but there are not price- and rental-vectors supporting it. In fact, by applying the Theorem 5.1 to the truncated problem

$$\max \int_0^{+\infty} e^{-\rho t} \log c_t dt$$

subject to

$$\dot{\tilde{\mathbf{s}}}_t^T = \tilde{x}_t \left[\mathbf{e}_2^T - (\delta_x - \delta_z)\, \mathbf{e}_1^T\right] - \delta_z \tilde{\mathbf{s}}_t^T - c_t \mathbf{e}_1^T$$

$$\tilde{s}_0^T = (h, k)\,, \qquad \tilde{x}_t \geq 0, \qquad \tilde{\mathbf{s}}_t^T \geq \tilde{x}_t \mathbf{e}_1^T, \qquad c_t \geq 0$$

it is easily checked that for $\rho \neq \delta_z - \delta_x$ (the case in which $\rho = \delta_z - \delta_x$ is slightly different)

$$c_t = \rho h e^{gt}, \qquad \mathbf{x}_t^T = h\mathbf{e}_2^T e^{gt}, \qquad \left(\tilde{x}_t = h e^{gt}\right)$$

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \left\{\left[k - \frac{h}{g + \delta_z}\right] e^{-\delta_z t} + \frac{h}{g + \delta_z} e^{gt}\right\} \mathbf{e}_2^T$$

$$\left(\tilde{\mathbf{s}}_t^T = \tilde{x}_t \mathbf{e}_1^T + \left\{\left[k - \frac{h}{g + \delta_z}\right] e^{-\delta_z t} + \frac{h}{g + \delta_z} e^{gt}\right\} \mathbf{e}_2^T\right)$$

with $g = -(\rho + \delta_x)$ is an optimal steady state solution of the original (truncated) problem and the price- and rental-vectors supporting the truncated problem are:

$$\tilde{\mathbf{v}}_t = \frac{1}{h\rho}\mathbf{e}_1 e^{-gt}$$

$$\tilde{\mathbf{q}}_t = \frac{(\delta_z - \delta_x)}{h\rho}\mathbf{e}_1 e^{-gt}.$$

It is easily checked that prices and rentals which are candidates to support the optimal solution of the original problem need to satisfy

$$\mathbf{v}_t = \frac{1}{h\rho}\mathbf{e}_1 e^{-gt} + y_t\mathbf{e}_3, \qquad \mathbf{q}_t = \frac{(\delta_z - \delta_x)}{h\rho}\mathbf{e}_1 e^{-gt} + w_t\mathbf{e}_3$$

where y_t and w_t need to satisfy the inequalities

$$(\delta_x - \delta_z)\, y_t + w_t \geq \frac{1}{h\rho}e^{-gt}, \qquad (1 - \delta_x + \delta_z)\, y_t \leq w_t$$

$$\dot{y}_t = (\rho + \delta_z)y_t - w_t, \qquad w_t \geq 0, \qquad y_t \geq 0.$$

Since the last but one inequality is redundant, it will be ignored here. By substituting

$$y_t = \theta_t e^{-gt}, \qquad w_t = \left[(\delta_z - \delta_x)\,\theta_t - \dot{\theta}\right] e^{-gt}$$

we get the system in θ_t

$$\dot{\theta}_t \leq -\frac{1}{h\rho}$$

$$\dot{\theta}_t \leq -\theta_t$$

$$\theta_t \geq 0$$

A function θ_t satisfying the first inequality is estimated from above by a decreasing straight line. Hence it cannot satisfy the third inequality for each t.

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